A STUDY OF THE ACCOMPLISHMENT OF HIGH SCHOOL
STUDENTS IN PLANE GEOMETRY AS SHOWN BY
RESULTS OF THE TEST USED IN THE
STATE HIGH SCHOOL GEOMETRY
CONTEST OF 1934

By

Doyle T. French

Contributions of the Graduate School
Indiana State Teachers College
Number 195

Submitted in Partial Fulfillment
of the Requirements for the
Master of Arts Degree
in Education

1935
The author wishes to acknowledge the helpful service of his thesis committee, Dr. Walter O. Shriner, Miss Inez Morris, Mr. Edward L. Abell. The author is indebted to Mr. Abell for assistance rendered in preparation for the study. The author is indebted to Miss Morris for reviewing the manuscript, for making helpful suggestions and for encouragement. The author is particularly indebted to Dr. Shriner for many useful and constructive criticisms.
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I. INTRODUCTION

A. Origin of Problem

The Mathematics Section of the Indiana State Teachers' Association sponsors a contest in mathematics each year. The committee in charge of the 1934 geometry contest proposed that a study be made of the test papers and that the results of such study be made available to the mathematics teachers of the state.

The mastery of plane geometry involves the acquisition of certain special abilities which are logically divided into four groups, the ability to acquire the necessary concepts, the ability to solve construction exercises, the ability to solve algebraic problems and the ability to give formal proofs. Each of these groups may be subdivided. It is evident that if we propose to study the accomplishment of students in geometry, tests must be constructed which measure the student's progress in each of these special abilities, and the results of such tests studied in the light of the type of ability each is designed to test. The examination used in this contest was constructed with this purpose in mind. It included concepts and definitions, practical applications, construction work and four methods of proof. The first problem of this study is to determine the progress students in general are making toward the acquisition of these abilities. This is an important issue since mathematics teachers must know definitely wherein they are failing and
and wherein they are succeeding if they hope to improve teaching methods.

There is an increasing demand that the subject matter taught in high school have a practical value. It must be of such a nature that it will function in life situations. The emphasis is changing from subject matter for the sake of knowing it, to subject matter that develops and teaches boys and girls in the art of thinking and living. It, therefore, is the method of thinking that is important rather than the memorization of facts. Geometry can meet this demand if greater emphasis is placed on the use of the important methods of formal proof and less emphasis on the reproduction of formal demonstrations given in the text book. To ascertain whether this shift of emphasis has actually been made or not the examination was constructed so that the student was obliged to use, or attempt to use, each of four methods of proof, namely, synthetic, analytic, algebraic and indirect. The second problem proposed in this study is to determine the extent to which students can use these four methods of proof.

In any course in mathematics, students make certain specific errors known as type errors. These errors may be due to the inherent nature of the work or to the method by which the material is taught. If the same error is made by several students it is evident that something is wrong with the instruction and that corrections must be made. The successful teacher will determine the cause of the repetition of the given error and attempt to remove the difficulty by remedial
teaching. However, much time and energy can be saved if the teacher knows in advance what errors are likely to occur and does the so-called "remedial teaching" in the first teaching. In this study we, therefore, propose to determine the errors that are likely to occur in plane geometry, in the hope that the knowledge of such errors will improve the teaching of the subject.

We feel that through knowledge of the degree to which students have acquired the specific abilities, the degree to which they can use the methods of proof, and the errors that are made, that certain needed changes in the teaching of geometry will be revealed.

The major problems proposed by this study are summarized and re-stated as follows:

1. To determine the order of accomplishment of pupils, entering the state geometry contest, in the four major objectives in geometry, namely, the ability to acquire the necessary concepts, the ability to solve the construction exercises, the ability to solve algebraic problems and the ability to give formal proofs.

2. To determine the extent to which these students understand, and show ability to use, the four methods of geometrical proof, namely, the synthetic method, analytic method, algebraic method and indirect method.

3. To determine types of errors made on the test.

4. To recommend needed changes in the teaching of
As revealed by the study.

B. Description of the Test

The examination was comprehensive in scope. It included the important types of work taught in plane geometry, such as concepts and definitions, practical applications, construction work and four methods of proof, namely, synthetic, algebraic, analytic and indirect. The test was of such length that no student could finish in two hours. Since it was to be used as a contest examination, it was necessary that it be constructed to test superior students. However, it contained only the fundamental types of work which should be taught in any well organized course in plane geometry.

The principal aims of the examination were, (1) to test the student's understanding of the fundamental concepts, (2) to test his ability to solve construction exercises, (3) to test his ability to apply algebraic methods to the solution of geometric problems, (4) to test his ability to give formal proofs.
II. REVIEW OF PREVIOUSLY WRITTEN MATERIAL

RELATING TO THE PROBLEM

A. General Objectives in the Teaching of Geometry

Before we can determine the accomplishment of students in plane geometry we must study the objectives to be reached, and then measure our progress by the degree to which these objectives have been attained. Quotations from men who have made definite contributions to the teaching of geometry are as follows:


"We should teach demonstrative geometry mainly as a course in reasoning and aim to develop powers and habits of careful, accurate and independent thinking rather than to present the subject as a finished model of deductive logic. We believe that geometry may be made the basis of study of the methods of reasoning which will be used in any field where the necessary facts are at hand."


"Our great aim in the tenth year is to teach the nature of deductive proof and to furnish pupils with a model for all their life thinking."


"One of the main purposes of geometry is to develop the power to analyze and reason."

"The real purpose of the subject is suggested more by the word 'demonstrative' than by the word 'geometry'. The chief purpose of this part of mathematics is to lead the pupil to understand what it is to demonstrate something, to prove a statement logically, to 'stand upon the vantage ground of truth' ."


"The purpose of teaching demonstrative geometry in the senior high school is not primarily to give the student information concerning the facts of space relationship.... The outstanding contribution of geometry, the element which has made it interesting to thinking men for 4000 years, the part which thrills children when it is correctly taught, is its logical structure, its organized reasoning with simple concepts, its inherent possibilities for producing in children the satisfaction of significant achievement. Geometry achieves its highest possibilities if, in addition to its direct and practical usefulness, it can establish a pattern of reasoning; if it can develop the power to think clearly in geometric situations and to use the same discrimination in non-geometric situations; if it can develop the power to generalize with caution from specific cases, and to realize the force and all-inclusiveness of deductive statements; if it can develop an appreciation of the place and function of
definitions and postulates in the proof of any conclusion, geometric or non-geometric; if it can develop an attitude of mind which tends always to analyze situations, to understand their inter-relationships, to question hasty conclusions, to express clearly, precisely, and accurately, non-geometric as well as geometric ideas."


Dr. Shibli made a study of the present aims of teaching geometry. He prepared a list of general aims compiled from various sources. Without attempting to classify these aims he sought to determine the relative importance of the aims by submitting the list to three hundred persons, one hundred leaders in the field of secondary mathematics who had contributed articles on the teaching of geometry to magazines, and two hundred students at summer school who had had experience as teachers of geometry. Each group was asked to select seven objectives which they consider most important and to list them in the order of importance. Dr. Shibli's conclusions were,

"Both studies show that there has been a marked trend away from preparation for college entrance examination, and towards preparation for actual life; away from formal discipline through the memorization of model proofs, and towards independent thinking and the power of discovery; away from making mathematicians and towards making intelligent citizens. The training of the mind is
still the supreme aim; but the method of attaining it has completely changed. Modern teachers no longer seek to train the mind through the knowledge of geometric facts and formal proofs; but through the mastery of processes and methods, and the cultivation of habits and ideals and powers that are effective in the life of the individual."

The objectives of teaching geometry given by the various authors in the field may be summarized in the following list:

1. To furnish a course in reasoning.
2. To develop powers and habits of careful, accurate and independent thinking.
3. To teach the nature of deductive proof and to furnish pupils with a model for all their life thinking.
4. To develop the power to analyze and reason.
5. To establish a pattern of reasoning.
6. To develop the power to think clearly.
7. To develop mental attitudes that are needed in life situations.
8. To develop the habit of clear and precise expression.

B. Definition of Each of the Four Methods of Proof

Dr. David Eugene Smith and Dr. William D. Reeve¹ make the distinction between the analytic and synthetic methods very clear in the following quotation, "Analysis and

synthesis are terms commonly used in chemistry. A chemist analyzes a substance and perhaps discovers that it contains iron, sulfur and oxygen. That is, he puts a sample in a test tube, a beaker or a retort, and subjects it to various processes in order to break it down into simpler compounds or elements. Synthesis in chemistry refers to putting together of elements or compounds by subjecting them to various processes in order to make some new desired product. Frequently, in fact usually, a synthetic process is suggested by and follows analysis. In other words analysis is a breaking down process used to discover something, and synthesis is a building up process based on analysis and used to produce a desired product. In geometry, analysis is the mental process of tearing down a geometric statement to discover the relationships upon which its truth or existence depends."

Dr. Arthur Schultze makes the following assertion, "Synthetic method leads from the known to the unknown, while the analytic method proceeds from the unknown to the known. In geometry a synthetic proof starts from the hypothesis and ends with the conclusion, while an analysis leads from the conclusion to the hypothesis. In a synthesis we say: A is true, therefore B is true and therefore C is true. In an analysis we reason: C is true if B is true and B is true if A is true. But A is true, hence C is true." "A synthesis shows that every step is true, but does not

explain why this step was taken. A synthetic proof convinces the reader that the fact to be demonstrated is true, but does not reveal to him the real plan of the demonstration, does not tell him why the sequence of arguments was selected. Proofs are not discovered by the synthetic method and if forgotten, synthetic demonstrations are most difficult to reconstruct. But synthetic proofs are usually short and elegant, and are in place when no pedagogical conditions need to be considered."

"An analysis, on the other hand, is lengthy and not elegant, but it is the only method that accounts fully for each step of the demonstration. It is the only method by which a student can hope to discover proofs, or to re-discover them after they are forgotten. Analysis is the method of discovery, synthesis is the method of concise and elegant presentation. Students in secondary schools should be made to discover demonstrations by analysis, but after this has been accomplished the proof may be represented synthetically."

Dr. Clifford Brewster Upton defines the indirect method of proof by saying, "When the truth of a proposition is established by showing that to assume its contradictory as true leads us to a conclusion which is known to be false, the proposition whose truth is thus demonstrated is said to be proved indirectly. The conclusion resulting from assuming the contradictory proposition true is considered false or absurd if it is inconsistent with something previously accepted."

as true, that is, if it contradicts the given data, some axiom or postulate, or some previously proved theorem."

The algebraic method of proof involves the use of algebraic equations. The geometric facts are expressed by the use of an equation, or equations. These are changed by the various axiomatic principles such as adding equals to equals, multiplying or dividing both sides of the equation by the same number, substituting equals for equals, solving for a given letter, and so on, until the final equation expresses the theorem which we proposed to prove.

C. Relative Importance of Each Method of Proof in the Thinking of the Individual After Completing the Course

The analytic and indirect methods of proof are the most important as tools, or methods of reasoning useful to the individual in actual life situations. In the analytic method he learns to examine all the known facts concerning a problem, to distinguish between important and irrelevant facts and reason step by step until he discovers a solution to the problem. These are the steps which must be followed in solving the many perplexing problems of life. As has been pointed out, the analytic method is a method of discovery while the synthetic method is a method of presentation. It is important that an individual be able to discover solutions for his problems and that he use these solutions to the betterment of his own happiness and the happiness of his associates. A training in logical processes is as necessary for the
average citizen as study in other fields of subject matter. He cannot draw valid conclusions in other fields unless he has had some training in logical processes. Geometry furnishes the best available material for this training because (1) it varies from the simplest to the most complex, (2) it starts with few assumptions and builds a body of established truths which can be used to establish still further truths, (3) the student is led to believe in reason and is made to feel the value of a demonstration, (4) the student realizes the possibilities of detecting fallacies and receives training in analysis of argument for such, (5) the authority of the propositions, which are established by his reasoning from the simple axioms, appeal to the student.

The main purpose of teaching the indirect method of proof is not to enable the student to understand the indirect proof encountered in his work in geometry but rather to equip him with a tool, a method of reasoning, applicable in definite life situations after leaving school. In life he will have many opportunities to use the indirect method. He needs to become so familiar with the method as practiced with geometrical materials that it will become a part of his reasoning process to the end that it will function in non-geometric situations.

The algebraic method may be used to a good advantage to introduce formal demonstrations. The students are familiar with the terms and processes used in algebra and therefore are able to understand a formal proof introduced by use of an
algebraic method. This serves to lead the pupil to a clearer conception of the new facts of geometry. The use of the algebraic method helps to unify algebra and geometry. Students are able to see the interrelationship of the subjects. It assists students in retaining the facts learned in algebra. It gives a practical application to algebra. Many theorems capable of proof by other methods, can be proven more readily by the algebraic method. Students and teachers like the method because it is definite. Many theorems necessary for proving certain other theorems involving analytic or synthetic methods can be proven by the algebraic method.

Until recently the synthetic method has received greater emphasis than the other methods. However, recently there has been a shift of emphasis from the synthetic method to the analytic and indirect methods. This shifting of the emphasis is not to the discredit of the synthetic method. It is important in that it teaches definite, clear presentation of the facts concerning a given situation. This ability is used to convince others of value of a project or theory. The synthetic method is the easiest since it is direct with known facts as basis. It is fundamental in that it is the method of presentation of the proof frequently discovered by some other method.

The analytic and synthetic methods are interrelated in that each is necessary to make the finished product. The analytic method is the method of discovery. It is the tool by which the student attacks the theorem or original exercise.

The student, through the logical reasoning of analysis
discovers a proof to a given exercise. After having discovered the proof, he must present his findings in clear, concise and orderly form. To do this he must have at his disposal the synthetic method. It is important that an individual after having discovered a truth, be able to convince others of his findings. Therefore, the analysis should be followed by a synthetic demonstration in every case. Each method is important but used for a different purpose. The analytic method is the method of discovery. The synthetic method is the method of presentation of the proof discovered.
III. VALIDITY OF THE TEST

A. Validity

Professor G. M. Ruch defines validity "as the degree to which the test parallels the actual flow of instruction, and of the care exercised in choosing important materials, in excluding non-essentials, and in producing a correct balance of material used in the examination." We shall examine this test and determine to what degree it measures up to the standard set out in Ruch's definition. The teachers of plane geometry throughout the state use *Modern Plane Geometry* by John R. Clark and Arthur S. Otis as the basic text. They are also required to follow rather closely the state course of study in plane geometry. Therefore it can be assumed that if this test contains material selected from these two sources, it will "parallel the actual flow of instruction."

Table I presents data showing the source from which the test material was selected. The second column gives the page in Clark-Otis *Modern Plane Geometry* on which the problem, exercise or theorem of each item of the test is treated. In many cases the same problem or theorem is used in the test as found in the text, while in other cases the facts which are necessary to the solution of the test element are given. The third column of the same table shows the page in the state course of study on which plans are given for the teaching of the material used in the various items. A study of this

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1G. M. Ruch, *The Objective or New-Type Examination* (Scott, Foresman and Company, 1929), p. 40.
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The table reveals that the materials used in the test have been carefully selected from the guides used by the teachers of the state. The test is, therefore, valid from the standpoint that it measures what it claims to measure.

Four types of material were included in the test, namely, thirty-five informational exercises, fifteen algebraic problems, nine construction exercises, and six formal proofs. Since these are the essential types of work taught in plane geometry, and since there is a sufficient sample of each used, it is evident that a correct balance of material was obtained.

**TABLE II**

**THE SOURCE FROM WHICH THE MATERIAL OF SECTION TWO OF THE TEST WAS SELECTED**

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<th>Test Item</th>
<th>Page on Which Same Material Is Given in State Adopted Text</th>
<th>Page on Which Same Material Is Outlined in State Course of Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>135</td>
<td>60,61</td>
</tr>
<tr>
<td>2</td>
<td>155,156</td>
<td>62,63</td>
</tr>
<tr>
<td>3</td>
<td>104</td>
<td>58,59</td>
</tr>
<tr>
<td>4</td>
<td>184</td>
<td>64,65</td>
</tr>
<tr>
<td>5</td>
<td>182</td>
<td>60,61</td>
</tr>
<tr>
<td>6</td>
<td>164</td>
<td>62,63</td>
</tr>
<tr>
<td>7</td>
<td>147,148</td>
<td>60,61</td>
</tr>
<tr>
<td>8</td>
<td>148</td>
<td>60,61</td>
</tr>
<tr>
<td>9</td>
<td>127</td>
<td>60,61</td>
</tr>
</tbody>
</table>

### TABLE IV

THE SOURCE FROM WHICH THE MATERIAL OF SECTION FOUR OF THE TEST WAS SELECTED

<table>
<thead>
<tr>
<th>Test Item</th>
<th>Page on Which Same Material Is Given in State Adopted Text</th>
<th>Page on Which Same Material Is Outlined in State Course of Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>155</td>
<td>62,63</td>
</tr>
<tr>
<td>2</td>
<td>67,114</td>
<td>56,57,58,59</td>
</tr>
<tr>
<td>3</td>
<td>64,96,168</td>
<td>56,57,58,59</td>
</tr>
<tr>
<td>4</td>
<td>69</td>
<td>56,57</td>
</tr>
<tr>
<td>5</td>
<td>80,86,100</td>
<td>56,57</td>
</tr>
<tr>
<td>6</td>
<td>185</td>
<td>64,65</td>
</tr>
</tbody>
</table>
B. Reliability

Professor G. M. Ruch\(^1\) makes the assertion that, "Reliability is guaranteed in two principal ways: (a) objectivity of scoring and (b) extended sampling (length of test)."

Section I consisted of multiple choice items, with four possible responses to each item. Each item was so constructed that only one of the responses was correct. Only one answer was correct for each of the fifteen problems of Section II. Keys,\(^2\) stating the exact answers to the items of the first and second sections, were furnished. This arrangement insured objectivity of scoring. The scoring of the other two sections was less objective. However, samples of correct constructions were furnished for Section III, and samples of correct formal proofs were furnished for section IV. A complete list of rules and regulations for grading papers covering the entire test was also furnished. It, therefore, is evident that every possible precaution was taken to make the scoring objective.

Sufficient evidence has already been given to show that the test was long enough to insure a good sampling of the material taught in plane geometry.

---

\(^1\)G. M. Ruch, *The Objective or New-Type Examination* (Scott, Foresman and Company, 1929), p. 62.

\(^2\)See Appendix, Part C.
IV. METHOD OF ANALYSIS OF TEST RESULTS

A. A Study of Responses Given to Each Item of Section I

Section I of the test consisted of thirty-five multiple choice items. The four possible responses to each item were listed as a, b, c, and d, respectively. Students were asked to place the letter representing the correct response in the space provided at the right-hand margin of the paper. Since the response which the student selected as the correct response was the only reaction we had concerning his thinking, it was necessary to confine our analysis to a study of the responses given to each item. This study was made by determining the per cent of students that gave "a" as the answer, per cent that gave "b" as the answer, per cent that gave "c" as the answer, per cent that gave "d" as the answer and the per cent omitted, for each of the thirty-five items. The items were ranked according to difficulty. The item answered correctly by the smallest per cent of students was considered the most difficult and given the rank of one. A study of the tabular data collected from this analysis was made to determine significant errors made by the group.

B. A Study of Answers Given Each Problem of Section II

Section II consisted of fifteen problems. Students were asked to place the answer, which they obtained by solving the problem, in the space provided at the right-hand margin of the page. They were not required to show their work. The answers were the only evidence we had of their thinking. It,
therefore, was necessary to confine the analysis of this section to the study of the answers. The per cent of correct answers to each of the fifteen problems was found. A list of incorrect answers which occurred frequently was made and the per cent of occurrence calculated. The per cent omitted was also determined. The problems were ranked according to difficulty. A study of the tabular data collected from this analysis was made to determine significant errors. To determine the errors in the solution of the problem the incorrect answers given to each problem were examined. Many times it was necessary to solve the problem incorrectly in order to arrive at the answer obtained by the student. This process usually revealed the error made. Consequently this technique made it possible to point out specifically the errors made in methods of obtaining the results.

C. A Study of Construction Given to Each Exercise of Section III

Section III required the drawing of the constructions for nine exercises. Space was provided for each drawing. This made possible a very careful study of each construction since all the work of the student was given. Each exercise was studied by (1) checking the constructions that were correct in all details, (2) checking the constructions that were correct in only certain details, (3) checking constructions wrong in all details. A classification of the details in which the constructions were wrong was made. From this classification certain errors were noted. By determining the per cent correct
in all details the exercises were ranked according to difficulty as in other sections.

D. A Study of Formal Proofs. Given to Each Theorem of Section IV

Section IV required the writing of six complete formal proofs. A statement of the theorem and the drawing required for the theorem were given in each exercise. The purpose of this was to secure uniformity of lettering and thus reduce the difficulties in reading and scoring the papers. Students were informed in the directions at the beginning of Section IV that the proof should be clear, concise and complete to be acceptable. They were required to use the analytic method in the third theorem and the indirect method in the fourth. No requirements were made concerning the method to be used in the other four theorems, however, they were lead to use the synthetic and algebraic methods by the nature of the theorems.

To ascertain the extent to which students understand and show ability to use the four methods of geometrical proof, a careful study was made of each proof given by the students. The proofs were classified into four classes, proof satisfactory, proof not satisfactory, proof omitted, and proof attempted by another method. If the student showed sufficient evidence of understanding the method involved and showed ability to use it, although some minor error was made, it was entered as proof satisfactory. If a student gave a good proof to a theorem by use of a method other than the one he was asked
to use, it was not entered as proof satisfactory but entered as proof attempted by another method. It is evident, for example, that if a student deliberately ignores the directions to use the analytic method and attempts the proof by the synthetic method, he does not understand the analytic method.

E. The Per Cent of Accomplishment on Each Section of the Test

Section I measured the student’s ability to acquire the necessary concepts, Section II measured the ability to solve problems, Section III measured the ability to solve construction exercises, and Section IV measured the ability to give formal proofs. In order to determine the order of accomplishment of pupils, entering the contest, in these four major abilities it was necessary to work out some type of accomplishment ratio for each section of the test. Since we had in the other analyses the percentage of correct responses to each item of the section, it was discovered that the average of these percentages represented the per cent of correct responses for the entire section. The average per cent of correct answers was calculated by this plan for each section of the test and used as a basis for comparison of the results on the four sections.
V. PRESENTATION OF DATA

The chief purpose of the analysis of Section I of the test was to ascertain the extent to which students know the fundamental concepts and definitions taught in plane geometry and also to discover significant errors. Since this section was composed of multiple choice items, each item was analyzed by determining the number of students who gave the "a" response, the number who gave the "b" response, the number who gave the "c" response, the number who gave the "d" response, and the number who omitted the item. For example in the item, "The locus of all points one inch distant from a circle with a 2-inch diameter is (a) a circle, (b) a line, (c) two circles, (d) a circle and a point," one hundred forty-one gave "a" as the correct response, three gave "b", eighty-six gave "c", thirty-eight gave "d", and three omitted it. In order to make comparisons these readings were changed to per cents by dividing each by two hundred seventy-one, the total number of papers. The results then read, fifty-two per cent "a", one per cent "b", thirty-two per cent "c", fourteen per cent "d", one per cent omitted.

The complete record of this analysis is given in the following table. The example just given is the first item listed in the table. The first column gives the exact copy of the item as it appeared in the test, the second column gives the rank of the item according to difficulty, the other columns of the table show the per cent of responses for each of the possible answers. The correct answer is indicated by a star in the column, the heading of which represents the correct response.
### TABLE V

**THE PER CENT OF OCCURRENCE OF EACH OF THE FOUR RESPONSES TO EACH ITEM OF SECTION ONE, GEOMETRIC CONCEPTS**

<table>
<thead>
<tr>
<th>Item</th>
<th>Rank</th>
<th>Per Cent of Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. The locus of all points 1 inch distant from a circle with a 2-inch diameter is (a) a circle, (b) a line (c) two circles, (d) a circle and a point.</td>
<td>1</td>
<td>52 132 *14 1</td>
</tr>
<tr>
<td>6. If two sides of one triangle are equal respectively to two sides of another triangle and the included angles are supplementary, the triangles are (a) congruent, (b) equilateral, (c) similar, (d) equal.</td>
<td>2</td>
<td>33 332 *26</td>
</tr>
<tr>
<td>33. Lines are concurrent if they (a) are everywhere equally distant, (b) intersect in one point, (c) lie in the same plane, (d) form a triangle...</td>
<td>3</td>
<td>20 *40 33 2 5</td>
</tr>
<tr>
<td>20. The number of lines required in drawing the altitudes, medians, and angle bisectors in an isosceles triangle is (a) five, (b) six, (c) seven, (d) nine.</td>
<td>4</td>
<td>24 15 *45 15 1</td>
</tr>
<tr>
<td>17. The number of points a given distance from two intersecting lines is (a) two, (b) four, (c) one, (d) any number.</td>
<td>5</td>
<td>11 *51 5 33</td>
</tr>
<tr>
<td>14. If an angle of a triangle is unchanged but each of the two including sides is doubled, the area is (a) two times, (b) three times, (c) four times, (d) five times as great.</td>
<td>6</td>
<td>20 21 *55 2 2</td>
</tr>
<tr>
<td>30. The locus of the hub of a wheel turning on a level straight-away is (a) two lines, (b) a circle, (c) a straight line, (d) several circles.</td>
<td>7</td>
<td>9 21 *61 7 2</td>
</tr>
</tbody>
</table>

*Per cent of correct responses.*
TABLE V. (Continued)

<table>
<thead>
<tr>
<th>Question</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>32. The area of the largest triangle that can be inscribed in a semicircle whose radius is ( r ) is (a) ( r^2 ), (b) ( r^3 ), (c) ( 2r^2 ), (d) ( 2r^3 ).................</td>
<td>8</td>
<td>*63</td>
<td>4</td>
<td>26</td>
</tr>
<tr>
<td>34. When the shadow of a flag pole is longer than the pole itself, the sun's rays strike the earth at an angle of (a) less than ( 45^\circ ), (b) ( 45^\circ ), (c) more than ( 45^\circ ), (d) ( 90^\circ )......</td>
<td>9.5</td>
<td>*68</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>11. The locus of the mid-points of parallel chords in a circle is (a) a circle, (b) tangents at the end points, (c) the perpendicular bisector of one of the chords, (c) the center of the circle.....................</td>
<td>9.5</td>
<td>13</td>
<td>4</td>
<td>*68</td>
</tr>
<tr>
<td>21. The median of a triangle divides the triangle into two triangles that are (a) isosceles, (b) equal, (c) congruent, (d) equilateral......</td>
<td>11</td>
<td>8</td>
<td>*71</td>
<td>16</td>
</tr>
<tr>
<td>10. An angle that is nine times as large as its complement is (a) ( 90^\circ ), (b) ( 80^\circ ), (c) ( 162^\circ ), (d) ( 81^\circ ).................</td>
<td>12</td>
<td>15</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>23. The acute angle formed by the hands of the clock at 2:00 P.M. is (a) ( 30^\circ ), (b) ( 45^\circ ), (c) ( 60^\circ ), (d) ( 90^\circ ).................</td>
<td>13</td>
<td>7</td>
<td>5</td>
<td>*76</td>
</tr>
<tr>
<td>16. The authors of the Indiana State adopted text in geometry are (a) Clark and Nyberg, (b) Strater and Upton, (c) Clark and Otis, (d) Nyberg and Otis.............</td>
<td>14</td>
<td>12</td>
<td>3</td>
<td>*78</td>
</tr>
<tr>
<td>26. If a proportion, ( \frac{m}{n} = \frac{v}{w} ), has been rearranged to read ( \frac{w}{v} = \frac{m}{n} ), it has been rearranged by (a) substitution, (b) inversion, (c) alternation, (d) subtraction.........</td>
<td>15</td>
<td>3</td>
<td>*80</td>
<td>16</td>
</tr>
<tr>
<td>24. The supplement of an angle of 86 degrees is (a) ( 4^\circ ), (b) ( 94^\circ ) (c) ( 104^\circ ), (d) ( 184^\circ ).............</td>
<td>16.5</td>
<td>5</td>
<td>*85</td>
<td>9</td>
</tr>
</tbody>
</table>

*Per cent of correct responses.
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2. In the proportion, ( m/x = x/y ), the ( x ) is called the</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(a) fourth proportional,</td>
<td>16.5</td>
<td>15 *85</td>
</tr>
<tr>
<td></td>
<td>(b) mean proportional,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c) third proportional,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(d) consequent..................</td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

8. If the proportion, \( 7/3 = 5/m \), has been rearranged to read \( 7/5 = 9/m \), it was rearranged by (a) inversion, (b) substitution, (c) subtraction, (d) alternation.................... | 19.5 | 7  | 4 | 1 *88 |

12. If two lines are cut by a transversal making the opposite interior angles equal, they are (a) oblique, (b) perpendicular, (c) parallel, (d) equal.......................... | 19.5 | 3  | 6 | *88  |

15. A proportion is defined as (a) ratio, (b) a statement regarding similar figures, (c) any fractional equation, (d) an equality of two ratios. | 19.5 | 7  | 3 | 2 *88 |

19. Three angles of one triangle equal respectively to three angles of another triangle make the triangles (c) congruent, (b) equal, (c) regular, (d) similar............... | 19.5 | 7  | 2 | 3 *88 |

22. The locus of all points a given distance from a given line segment is (a) two parallel lines, (b) two semicircles, (c) two circles, (d) two parallel line segments and two semicircles.... | 22.5 | *90 | 9 | 1 |

31. The number of triangles that can be constructed on a fixed base, MN, with an altitude of six inches is (a) one, (b) two, (c) none, (d) indefinite number.................. | 22.5 | 4  | 6 | *90 |

1. The locus of points equidistant from a given point is (a) line, (b) a circle, (c) an angle, (d) a point.... | 24.5 | 5  | *91 | 4 |

7. The complement of an angle of \( n \) degrees is equal to (a) \( 90^\circ - n \), (b) \( 180^\circ - n \), (c) \( n - 90^\circ \), (d) \( n 160^\circ \)...... | 24.5 | *91 | 4  | 2  | 3   |

*Per cent of correct responses.
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4. Two angles of any right triangle are</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) supplementary,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) complementary,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) equal,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) obtuse</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>5</td>
<td>92</td>
</tr>
<tr>
<td><strong>13. If two lines in a plane are perpendicular to the same line,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>they are</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) parallel,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) equal,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) bisected,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) perpendicular, to each other</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>27</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td><strong>25. The sum of the interior angles of a polygon equals</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) two straight angles,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) two right angles,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) ((n - 2)) right angles,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) ((n - 2)) straight angles</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>28.5</td>
<td>3</td>
<td>94</td>
</tr>
<tr>
<td><strong>28. An inscribed angle is formed by</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) two radii,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) two chords intersecting within the circle,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) two secants intersecting outside the circle,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) two chords intersecting on the circumference</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>28.5</td>
<td>5</td>
<td>94</td>
</tr>
<tr>
<td><strong>27. A polygon with six equal sides is called a</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) decagon,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) octagon,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) pentagon,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) hexagon</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>1</td>
<td>96</td>
</tr>
<tr>
<td><strong>35. A circle may be inscribed in any</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) triangle,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) quadrilateral,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) pentagon,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) hexagon</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>31.5</td>
<td>1</td>
<td>97</td>
</tr>
<tr>
<td><strong>3. The area of any parallelogram is equal to</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) the sum of the sides,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) the product of the two sides,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) the product of its base and altitude,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) half the product of its base and altitude</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>31.5</td>
<td>97</td>
<td>3</td>
</tr>
<tr>
<td><strong>9. The sum of all the angles formed on one side of a line at a given point on the line is</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(a) a straight angle,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(b) an acute angle,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(c) a right angle,</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(d) a 360° angle</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>33.5</td>
<td>98</td>
<td>2</td>
</tr>
</tbody>
</table>

*Per cent of correct responses.*
On the average 78.8% of the responses to Section I were correct. This percentage is gratifying because it shows that students in general have a fair understanding of the geometric concepts and definitions. However, there are certain outstanding errors revealed by the preceding table which are worthy of mention. In the first item it will be noted that only fourteen per cent gave the correct response, "a circle and a point," while fifty-two per cent gave "a circle." One hundred forty-one students or fifty-two per cent did not comprehend that, since the locus was an inch distant from the circle, the point, the center of the circle satisfies the conditions of the locus just as the points on the outer circle. In locus problems students frequently do not comprehend that all points that fulfill the given condition lie on the locus. There is a tendency to omit part of the locus.

In the item, "If two sides of one triangle are equal respectively to two sides of another triangle and the included angles are supplementary, the triangles are (a) congruent, (b) equilateral, (c) similar, (d) equal," thirty-
three per cent gave "congruent" as the answer. In all probability these students did not read the item in its entirety. Since the first part of the hypothesis is similar to the hypothesis of the familiar congruency theorem, they assumed it to be the congruency theorem and hastened to check the answer. This led to the response "congruent." These eighty-nine students failed to check the hypothesis accurately.

One hundred sixty-three students did not answer correctly the item, "Lines are concurrent if they (a) are everywhere equally distant, (b) intersect in a point, (c) lie in the same plane, (d) form a triangle." Twenty per cent gave "everywhere equally distant," thirty-three per cent gave "lie in the same plane," two per cent gave "form a triangle," while five per cent omitted the item. It is evident from the distribution of the answers that these students lacked the knowledge of the meaning of the word concurrent. Students experience difficulty with certain of the less common definitions.

An opposite situation existed for the item, "The number of points a given distance from two intersecting lines is (a) two, (b) four, (c) one, (d) any number," in which the per cent of responses are respectively, eleven, fifty-one, five, and thirty-three. It will be noted that the incorrect answer, "any number," occurred frequently. These eighty-nine students probably substituted the word equal for the word given, thus reading the item, "points equally distant from," instead of, "points a given distance from." This is another
case in which students failed to check the hypothesis accurately. The twenty-nine who gave "two" failed to see the points located on the bisector of the other vertical angles. Fourteen students saw only one point. This shows a tendency for students to omit part of the locus.

The common errors revealed by the analysis of Section I are summarized as follows:

1. In locus problems students do not comprehend that all points that fulfill the given condition lie on the locus. There is a tendency to omit part of the locus.

2. Students fail to check carefully enough to see if all points which satisfy the given condition are included in the locus described.

3. Evidence shows a lack of accurate check on hypothesis.

4. There is poor knowledge of the less common definitions.

5. There is a tendency to omit part of conclusion when a part has been proven.

6. Conclusions are reached too hastily. Students lack the ability to sketch a problem as a means to a solution.

7. Students read into the problem a hypothesis that is not present.

8. Students frequently fail to comprehend the meaning of the word locus.
The purpose of the analysis of Section II was to determine the ability of students to apply computational methods to the solution of geometric problems. This section of the test consisted of fifteen geometrical problems each of which could be solved by algebra and arithmetic if the student knew the geometric law or fact on which to base his solution. Each problem was analyzed by checking and listing all the answers given by the two hundred seventy-one students. From this list the number who gave the correct answer was determined, also the frequency with which the incorrect answers occurred. For example in the problem, "If the circumference of a circle is increased by 14 inches, by how much is the radius increased? Express the answer in terms of \( \pi \)".

Thirty-five students gave the correct answer, \( \frac{14}{2\pi} \). A study of the list of incorrect answers revealed that the answer \( \frac{14}{\pi} \) occurred sixteen times, the answer \( \frac{14\pi}{4} \) occurred six times and that eighty-nine other answers were given. One hundred twenty-four did not give an answer to the problem. These tabulations were changed to per cents by dividing the frequency of each by two hundred seventy-one, the total number of papers. The results of the analysis of this problem then read, thirteen per cent gave the correct answer \( \frac{14}{2\pi} \), six per cent gave the incorrect answer \( \frac{14}{\pi} \), two per cent gave the incorrect answer \( \frac{14\pi}{4} \), thirty-three per cent gave other answers, forty-six per cent did not give an answer.

The complete record of this analysis is given in Table III. The illustration given above is the first problem
<table>
<thead>
<tr>
<th>Problem</th>
<th>Rank</th>
<th>Answers Given and Per Cent of Frequency</th>
<th>Per Cent of other Answers</th>
<th>Per Cent Omitted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>An.</td>
<td>%</td>
<td>An.</td>
</tr>
<tr>
<td>10. If the circumference of a circle is increased by 14 inches, by how much is the radius increased? Express answer in terms of $\pi$...</td>
<td>1</td>
<td>*$14/\pi$</td>
<td>13</td>
<td>14/$\pi$</td>
</tr>
<tr>
<td>7. The sides of a triangle are 10, 12, and 16 inches, respectively. How long is the altitude to the longest side?.................</td>
<td>2</td>
<td>*7.5</td>
<td>19</td>
<td>8</td>
</tr>
<tr>
<td>3. A secant and a tangent drawn to a circle from an external point form an angle of 20°. If one arc intercepted on the circle is 70°, how large is the other arc?.....</td>
<td>3</td>
<td>*$110$</td>
<td>42</td>
<td>---</td>
</tr>
<tr>
<td>13. If the altitude of an equilateral triangle is 18 inches, how long is the side of the triangle?</td>
<td>4</td>
<td>*$12 \sqrt{3}$</td>
<td>48</td>
<td>6$\sqrt{3}$</td>
</tr>
<tr>
<td>2. If the acute angles of a right triangle are each 45° and one side including the right angle is b, what is the hypotenuse in terms of the side b?.........</td>
<td>5.5</td>
<td>*$b \sqrt{2}$</td>
<td>55</td>
<td>2b$^2$</td>
</tr>
<tr>
<td>14. One angle of a rhombus is 120°. If the shorter diagonal is 12 inches, find the side of the rhombus?.....................</td>
<td>5.5</td>
<td>*12</td>
<td>55</td>
<td>24</td>
</tr>
</tbody>
</table>

*Correct Answers.*
<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>TABLE VI.</strong> (Continued)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. How far from the center of a circle with a radius of 13 inches is a chord 24 inches long?</td>
<td>7.5</td>
<td>*5</td>
<td>57</td>
<td>2</td>
<td>10</td>
<td>1</td>
<td>3</td>
<td>16</td>
</tr>
<tr>
<td>12. What is the area of a square inscribed in a circle whose radius is 6 inches?</td>
<td>7.5</td>
<td>*72</td>
<td>57</td>
<td>36</td>
<td>6</td>
<td>144</td>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>8. If one side of a square is a, find the length of the diagonal in terms of a?</td>
<td>9</td>
<td>*s√2</td>
<td>67</td>
<td>√2s²</td>
<td>4</td>
<td>2s²</td>
<td>3</td>
<td>22</td>
</tr>
<tr>
<td>15. One angle of a triangle is twice the second and the third is five times the second. How large is each angle?</td>
<td>112.5</td>
<td>13 11/13</td>
<td>2x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. The bases of a trapezoid are 16 and 27 inches, respectively. How long is the line joining the mid-points of the non-parallel sides?</td>
<td>10.5</td>
<td>*22.5</td>
<td>79 136 2/13</td>
<td>4</td>
<td>y</td>
<td>1</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>11. One angle of a right triangle is 30°. How long is the hypotenuse in terms of the shorter leg, b, of the triangle?</td>
<td>12</td>
<td>*2b</td>
<td>82</td>
<td>b²</td>
<td>1</td>
<td>3b</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>9. The base of a triangle is 20 inches. How long is the segment joining the mid-points of the other two sides?</td>
<td>13</td>
<td>*10</td>
<td>85</td>
<td>15</td>
<td>1</td>
<td>13 1/3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4. If the vertex angle of an isosceles triangle is three times as large as the base angle, how many degrees in each angle of the triangle?</td>
<td>108</td>
<td>25 5/7</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. If two angles are complementary and one is 10° larger than the other, how large is each angle?</td>
<td>40</td>
<td>85</td>
<td>85</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Correct Answers.*
listed in the table. The first column of the table gives the problem as it appeared in the test, the second column the rank according to difficulty. The first problem listed was considered the most difficult and given the rank of one since it was answered correctly by the smallest number of the students. The last problem in the table was answered correctly by the largest number of students and therefore was considered the easiest and given a rank of fifteen. The other problems range between these according to the per cent of correct answers. The remainder of the table shows the "Answers Given and Per Cent of Frequency," "Per Cent of Other Answers" and "Per Cent Omitted."

A study of the incorrect answers of frequent occurrence will reveal certain difficulties which the student experienced in solving the problems. The six per cent who obtained the answer, \( \frac{14}{\pi} \), to the first problem in the table very likely confused the formulas for area of a circle and circumference of a circle. If the formula, \( \pi r \) is used, the solution to the equation, \( \pi r = 14 \) is \( \frac{14}{\pi} \). This is not a difficulty experienced by a large number but it is one which teachers must anticipate. Since in this same problem, forty-six per cent omitted it and thirty-three gave other answers, it is evident that the most general difficulty experienced was the lack of the knowledge of how to set up the equations in the two situations and compare them. The
correct solution is as follows:

<table>
<thead>
<tr>
<th>First Circle</th>
<th>Second Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = 2\pi r$</td>
<td>$C + 14 = 2\pi r_1$</td>
</tr>
<tr>
<td>$r = \frac{C}{2\pi}$</td>
<td>$r_1 = \frac{C}{2\pi} + \frac{14}{2\pi}$</td>
</tr>
</tbody>
</table>

Comparing these values of $r$ and $r_1$ we see that the increase in the radius is $\frac{14}{2\pi}$.

In the problem, "If the acute angles of a right triangle are each 45° and one side including the right angle is $b$, what is the hypotenuse in terms of the side $b$?", eight per cent gave $2b^2$ as the answer. These twenty students solved the problem correctly to the last step but forgot to find the square root of $2b^2$. This type of error is due to careless and hasty methods of work. These students no doubt reasoned the square on one side is $b^2$, the square on the other side is $b^2$, the sum of the two is $2b^2$, hence the hypotenuse is $2b^2$. If they had been taught to organize their work by using complete statements for each step, such errors would not occur. For example had they written $b^2 + b^2 = c^2$; $c^2 = 2b^2$; $c = b\sqrt{2}$; the presence of $c^2$ would have called attention to the fact that the hypotenuse is the square root of the sum of the squares on the other two sides.

It is interesting to note the peculiar reasoning of students when they are not sure of the solution. In the problem, "How far from the center of a circle with a radius of 13 inches is a chord 24 inches long?", ten per cent gave 2 as the answer. They compared the chord with the diameter, hence 25 minus 24 is 2. Three per cent gave one as the
answer. They no doubt subtracted 12 from 13. These students probably knew that they were wrong but wanted to show that they had done something. This is a method of self-defense. It may have been cultivated by teachers' giving credit for efforts or attempted solutions even though the work is incorrect.

The problem, "If two angles are complementary and one is $10^\circ$ larger than the other, how large is each angle?", was solved correctly by ninety-one per cent of the group yet three percent or eight students substituted the word supplementary for the word complementary and obtained the answers 85 and 95. There is often a confusion of words with similar meanings.

In the problem, "One angle of a triangle is twice the second and the third is five times the second. How large is each angle?", four per cent obtained the answers $13 \frac{11}{13}$, $27 \frac{9}{13}$, 138 $\frac{2}{13}$. These answers are correct if $X$ is the first angle, $2X$ the second and $10X$ the third. However, this is not the correct representation of the angles, because the problem states that one angle is twice the second, hence $X$ is the second angle and $2X$ the first. The third angle, which is five times the second, therefore is $5X$. This type of error is caused by one of two things--the student either did not read the problem carefully enough, or failed to write the preliminary statements of the problem such as, let $X$ equal the second angle, $2X$ equal first angle, $5X$ equal the third angle. We need to teach students to be more explicit
in their solutions. Many such errors can be avoided if all details of the solution are written on paper.

On the average, sixty-one per cent of the answers to the fifteen problems were correct. This may represent an accomplishment ratio for this section of the test. It can be noted at this point that since the average of section one was seventy-eight, students show lower ability in problem solution than in mastery of concepts and definitions.

Summary of type errors revealed by the above analysis:
1. Students frequently forget minor details in the solution of a problem.
2. There is a tendency to miss problems which involve difficult algebraic equations.
3. Although the theorem of Pythagoras is used frequently, students lack the ability to use it as a means to the solution of the more difficult problems.
4. As a means of self defense, students pretend to have thought through a problem although they are conscious that the solution is wrong. The assigning of credit to work that is incorrect encourages students to do this.
5. Frequently errors are made by attempting to omit certain important steps.

The section of the test devoted to construction was analyzed by examining each construction. The constructions were classified into three groups, namely, correct in all details, correct in only certain details and wrong in all details. The frequency in each group was changed to per
cent by dividing by two hundred seventy-one. For example, the exercise, "Construct the locus of vertex A of a right triangle having BC as hypotenuse," ninety-eight constructions or thirty-eight per cent were correct in all details, one hundred eleven or forty-one per cent correct in only certain details, and sixty-two or twenty-three per cent wrong in all details.

The complete record of this analysis is given in Table VII. The exercises were ranked as in other sections. The footnotes explain the particular detail in which the construction was wrong.

A study of the data given in Table VII reveals the fact that a significant error made in construction exercises is the omission of parts of the construction. For example in the most difficult exercise, "Construct the locus of vertex A of a right triangle having BC as hypotenuse," thirty-six per cent were correct in all details and forty-one per cent correct in only certain details. Since the forty-one per cent who drew a semicircle instead of a circle knew the fundamental theorem involved in the construction and was able to apply it to the solution of the exercise, we can see that seventy-seven per cent, thirty-six plus forty-one, knew how to make the construction. The same is true of the exercise, "Construct the locus of points at distant d from the circle whose center is M." A high percentage knew how to make the construction, although fifty-two per cent omitted part of the construction. Other illustrations may be cited to verify
TABLE VII
PER CENT OF CORRECT AND INCORRECT RESPONSES
TO SECTION THREE, CONSTRUCTION EXERCISES

<table>
<thead>
<tr>
<th>Problem</th>
<th>Rank</th>
<th>Per Cent Correct in all Details</th>
<th>Per Cent Correct in only Certain Details</th>
<th>Per Cent Wrong in all Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Construct the locus of vertex A of a right triangle having BC as hypotenuse</td>
<td>1</td>
<td>36</td>
<td>41</td>
<td>23</td>
</tr>
<tr>
<td>9. Construct the locus of points at distance d from the circle whose center is M</td>
<td>2</td>
<td>38</td>
<td>52</td>
<td>10</td>
</tr>
<tr>
<td>6. Construct the locus of the midpoints of all chords, in circle P, equal to XY. Describe steps taken</td>
<td>3.5</td>
<td>45</td>
<td>10</td>
<td>45</td>
</tr>
<tr>
<td>7. Construct a circle, outside of triangle ANG, and tangent to the three lines AR, AL, and NG</td>
<td>3.5</td>
<td>45</td>
<td>10</td>
<td>55</td>
</tr>
<tr>
<td>8. Construct a line that shall be tangent to circle C and perpendicular to line l</td>
<td>5</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>4. Construct the locus of the vertices of triangles above AB having AB as the base and area equal to the area of ABM</td>
<td>6</td>
<td>69</td>
<td>31</td>
<td>28</td>
</tr>
<tr>
<td>3. Divide the right angle AMR into three equal parts</td>
<td>7</td>
<td>72</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. One hundred twelve students, or 41%, constructed a semicircle having BC as a diameter. There were cases in which this semicircle was drawn above the line, and others in which it was drawn below the line. This group did not comprehend the fact that for the locus to include all the points which satisfy the condition, it must include the whole circle.

2. One hundred thirty-nine drew the outer circle but did not draw the inner circle. Two drew the inner circle but did not draw the outer circle. This represents 52% who did not draw the complete locus.

3. Twenty-six, or 10%, made the construction but were unable to describe the steps taken in making the construction.
TABLE VII. (Continued)

| 5. Construct two circles tangent internally and with radii equal to $r$ and $r'$... | 8 | 75 | 15 | 10 |
| 1. Locate a point which is equidistant from the three points, $A$, $R$, and $S$........ | 9 | 96 | 4 |

* Forty, or 15%, drew two circles tangent externally instead of internally.

This conclusion. There is a definite tendency for students to omit parts of the construction, although they know the theorems that are involved and the fundamental steps to be taken.

The tendency for students to confuse the meaning of similar words is again shown in the results found in the exercise, "Construct two circles tangent internally and with radii equal to $r$ and $r'$." No doubt those who drew the circles tangent externally could have drawn the circles tangent internally had they not confused the meaning of the two words.

Students experience some difficulty in describing the steps taken even though they can make the construction. This is shown by the fact that in the exercise, "Construct the locus of midpoints of all chords in circle $P$, equal to $XY$. Describe the steps taken," ten per cent who made the construction were unable to describe what they did.

62.9% of all the constructions were correct in all details. Therefore, 62.9% represents the accomplishment ratio for this section of the test.

The first theorem of Section IV was, "An angle inscribed
in a circle is equal in degrees to half the intercepted arc." This is theorem 239, page 155 in the state adopted text. The students were more or less familiar with this theorem since they had studied it as one of the formal demonstrations in the book. All those who attempted the proof used the synthetic method. Eighty-one or thirty per cent of the proofs were satisfactory. One hundred seventy-nine or sixty-six per cent of the proofs were not satisfactory. Eleven students omitted the theorem. This record is given in Table VIII. It is entered under synthetic method as exercise one. The fifth theorem, "The bisector of the four angles of a rectangle form a square," was also proven by use of the synthetic method. This theorem was taken from a list of review exercises found in the text. Since this may or may not have been solved in class the students no doubt were less familiar with it than with the first one. However, thirty-four per cent of the proofs were satisfactory. This is a slight increase over the per cent correct in the first one. Sixty per cent of the proofs were not satisfactory and five per cent were omitted. The averages of these results are given in the third column of Table VIII.

The second and sixth theorems were proven by the use of the algebraic method. Students were lead to use this method because of the nature of the theorems. The second theorem was, "Prove that any median of a triangle is less than half the perimeter." A drawing of a triangle with one median was given. The two segments of the side to which the median was
drawn were each labeled $\frac{1}{2}c$, the other sides $a$ and $b$ respectively, the median $m$. This arrangement made it very easy for the student to write the two inequalities,

\[ m < a + \frac{1}{2}c \]
\[ m < b + \frac{1}{2}c \]

By finding the sum of the two inequalities and dividing through by 2 the following inequality was obtained.

\[ m < \frac{1}{2}(a + b + c) \]

Only twenty-eight of the students were able to do this. Sixty per cent of the proofs were not satisfactory. Nine per cent omitted the theorem. The other theorem involving the algebraic method was, "The area of a trapezoid is equal to half the product of the altitude and the sum of the bases."

This is theorem 274, page 185 in the state adopted text. It is one with which they were very familiar, having worked it as a class demonstration in plane geometry. Fifty-two per cent of the proofs were satisfactory, forty-one per cent not satisfactory and nine per cent omitted.

The fourth theorem, "Two lines perpendicular to the same line are parallel," required the use of the indirect method. This is corollary lll, page 69 in the state adopted text, and is one with which they should have been familiar. However, only twenty-five per cent of the proofs were satisfactory. Seventy per cent were not satisfactory. Five per cent omitted the theorem.

The third theorem, "If two medians of a triangle are equal, the triangle is isosceles," required the use of the
analytic method. The following information was given to this theorem:

Statement of theorem: If two medians of a triangle are equal, the triangle is isosceles.

Given triangle ABC with EX and CY medians to AC and AB, respectively.

To prove that triangle ABC is isosceles.

State the steps you would use for an analytic proof.

Proof:

1. ABC is isosceles if I can prove AB = AC, since an isosceles triangle has two equal sides by definition.
2. 

Only thirty students, or eleven per cent, were able to give a satisfactory proof to this theorem by the analytic method. Ten per cent ignored the directions and the form of ruling, and attempted to use the synthetic method. Sixty per cent attempted to use the analytic method but were unable to work out a satisfactory proof. Nineteen per cent omitted the theorem.

A study of Table VIII shows that the algebraic method is best understood. The synthetic method ranks just a little below the algebraic method. However, since the sixth theorem was an easy theorem and one with which they were very familiar, having studied it just prior to the examination, and furthermore, since the results on the other theorem, which involved the algebraic method, are much lower, we are probably justified in saying that the algebraic and synthetic
methods are equally well understood. It is significant that the algebraic method is understood by a large group of students. It means that a definite effort is being made to tie up the work of algebra and geometry. The method of giving formal proofs can be developed directly from some simple algebraic statements. The results of this new angle of approach is shown here by the fact that students are able to use the algebraic method.

Since the synthetic method is used in many demonstrations in the book, we might expect students to show greater ability to use it than other methods less frequently used in the text. However, this tendency is not so noticeable since students show only a fair degree of ability to use the synthetic method.

The indirect method occurs here in third place. It was pointed out earlier in the discussion that the indirect
method of reasoning is important in that we discover the solution to many problems of life by indirect reasoning. We find only one-fourth of the students entering this contest are able to write a satisfactory indirect proof. Greater emphasis must be placed on the teaching of the indirect method.

Students show little ability to use the analytic method. It is evident from analysis of these test papers that this method is not receiving much emphasis. It is disheartening to know that our intelligent and capable students are not able to use the analytic method, the method that is so fundamental to the analysis of life situations and to the discovering of solution of problems later rearranged in synthetic form. In life we reach our conclusions largely by analytical reasoning. This type of reasoning is developed by the analytic method of proof. The best opportunity to teach it is in the development of original proofs of exercises.

The degree to which students show ability to use each of these four methods of formal proof is also shown by Figure 1.

Table IX gives the record of the same type of analysis as Table VIII except that it represents only fifty of the best papers. It should be noted that although the percentages of satisfactory proofs are somewhat higher, the four methods hold about the same order.

Table X shows the order of accomplishment of students in the four major objectives in geometry, namely, the
Figure 1. The comparison of the ability of students to use the methods of formal proof.
TABLE IX

SECTION FOUR, FORMAL PROOFS. THE RESULTS ON THE FIFTY BEST PAPERS

<table>
<thead>
<tr>
<th>Method of Proof</th>
<th>Synthetic</th>
<th>Algebraic</th>
<th>Indirect</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof Satisfactory</td>
<td>1 5 1&amp;5 6 2 6&amp;2 4</td>
<td>36%</td>
<td>72%</td>
<td>62%</td>
</tr>
<tr>
<td>Proof not Satisfactory</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Omitted</td>
<td>4 2 2 1</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proof Attempted by Another Method</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ability to acquire the necessary concepts, the ability to solve construction exercises, the ability to solve algebraic problems and the ability to give formal proofs. The per cent of correct responses for each section of the test is recorded in the second column of the table. The relationships are

TABLE X

PER CENT OF ACCOMPLISHMENT ON EACH OF THE FOUR SECTIONS OF THE TEST

<table>
<thead>
<tr>
<th>Section of the Test</th>
<th>Per Cent of Correct Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>I Informational Exercises</td>
<td>78.8</td>
</tr>
<tr>
<td>II Problem Solving</td>
<td>61</td>
</tr>
<tr>
<td>III Construction Exercises</td>
<td>62.9</td>
</tr>
<tr>
<td>IV Formal Proofs</td>
<td>30</td>
</tr>
</tbody>
</table>
also shown by Figure 2.

Table X reveals the fact that students are high in their ability to acquire the fundamental concepts and definitions of geometry. They are somewhat lower in their ability to solve problems and do construction exercises. It is gratifying to find that students know the geometric concepts, because success in all other phases of geometry depend on knowledge of definitions, axioms, theorems, corollaries and so on. We still are confronted with the problem in geometry, as in other branches of study, of teaching students to use the facts and information which they have acquired. If the fact is to be used in a new situation, such as the solution of a given problem, the student is confused. Evidence shows that students know the facts but are unable to use and apply them in new situations.

Students are very low in their ability to give formal proofs. It is discouraging to know that on the average only thirty per cent of our better students are able to write a satisfactory formal proof. It will have to be admitted that since the formal proofs were the last part of the test that the poor results on this section may be due somewhat to the lack of time. After all, how are we going to teach the second and fourth objectives as set out in our state course of study, namely, "To inculcate ideals of precision of thought and statement" and "To analyze a complex situation and recognize logical relations in geometry," if we do not teach formal proof? We cannot expect to maintain geometry
Figure 2. The comparison of the accomplishment of students in the four major objectives in geometry.
in the curriculum if we do not reach the major objectives
on which we base our defense. The mathematics teacher must
rally to the challenge and resolve to teach formal proofs
with a new interest and enthusiasm.
VI. SUMMARY AND CONCLUSION

The accomplishment of pupils, entering the state geometry contest in the four major objectives in geometry assume the following order: (1) ability to acquire the necessary concepts, (2) ability to solve construction exercises, (3) ability to solve algebraic problems, (4) ability to give formal proofs. Students are high in their ability to acquire the concepts and definitions. It is pleasing to find that students know the geometric concepts because success in all other phases of geometry depend on knowledge of definitions, axioms, theorems, corollaries and so on. We must strive to maintain the high standard of achievement already reached in this phase of geometry. Students are somewhat lower in ability to solve problems and to do construction exercises. Evidence indicates that students know the geometric facts but are unable to use and apply them in new situations. Greater emphasis should be placed on the teaching of problems and construction exercises. Students are very low in their ability to give formal proofs. Some further study should be made to ascertain the cause of the partial failure in this phase of geometry. It may be possible that we are expecting students to work out so many proofs that the whole thing leads to confusion rather than the mastery of the method of proof. After all it is the method of proof that is important and not the theorem. It is the method or reasoning that they are going to retain and use in life.

The algebraic and synthetic methods are better understood
than the indirect and analytic method. Although the teaching of formal proofs in general should be stressed, the teaching of the indirect and analytic methods need very special emphasis. The indirect method is important in that we discover the solution to many problems of life by indirect reasoning. Students show very little ability to use the analytic method. Evidence indicates that this method is not receiving much emphasis. In life we reach our conclusions largely by analytical reasoning. This type of reasoning is developed by the analytic method of proof. Since it is a method of discovery it can be taught as a means to the solution of original exercises.

The significant errors revealed by the analysis are as follows:

1. In locus problems students do not comprehend that all points that fulfill the given condition lie on the locus. There is a tendency to omit part of the locus.
2. Evidence shows a lack of accurate check on hypothesis.
3. There is poor knowledge of the less common definitions.
4. Conclusions are reached too hastily. Students lack the ability to sketch a problem as a means to a solution.
5. Students frequently forget minor details in the solution of a problem.
6. There is a tendency to miss problems which involve difficult algebraic equations.
7. Frequently errors are made by attempting to omit
certain important steps.

8. Although the theorem of Pythagoras is used frequently, students lack the ability to use it as a means to the solution of the more difficult problems.

9. As a means of self defense, students pretend to have thought through a problem although they are conscious that the solution is wrong. The assigning of credit to work that is incorrect encourages students to do this.

10. There is a tendency for students to omit parts of the construction, although they know the theorems that are involved and the fundamental steps to be taken.

11. Students experience difficulty in describing the steps taken even though they can make the construction.
VII. APPENDIX

A. Bibliography


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This examination was prepared by a committee of geometry teachers. It was devised to require a testing period of two hours. While the committee was instructed to prepare a test which no examinee could complete in two hours, this examination is one of power rather than speed.

Students taking this examination should be equipped with a straight edge, compasses, and one or two well-sharpened pencils. There is no objection to an examinee being supplied with several sheets of blank paper for use in making some experimental analyses.

PART I (each item points)

Directions: Select the one word or group of words that makes the statement true. Draw a line under the right answer and then put the number of that answer in the space at the right.

EXAMPLE:
A triangle is determined by (1) one point, (2) three points, (3) two points, (4) four points. 3

1. The locus of points equidistant from a given point is (1) a line, (2) a circle, (3) an angle, (4) a point 1

2. In the proportion, \( \frac{a}{b} = \frac{c}{b} \), the \( c \) is called the (1) fourth proportional, (2) mean proportional, (3) the third proportional, (4) consequent 2

3. The area of any parallelogram is equal to (1) the sum of the sides, (2) the product of two sides, (3) the product of its base and altitude, (4) half the product of its base and altitude 3

4. Two angles of any right triangle are (1) supplementary, (2) complementary, (3) equal, (4) obtuse 4

5. The locus of all points 1 inch distant from a circle with a 2 inch diameter is (1) a circle, (2) a line, (3) two circles, (4) a circle and a point 5

6. If two sides of one triangle are equal respectively to two sides of another triangle and the included angles are supplementary, the triangles are (1) congruent, (2) equilateral, (3) similar, (4) equal 6
7. The complement of an angle of \( n \) degrees is equal to 
   (1) \( 90^\circ - n \), (2) \( 180^\circ - n \), (3) \( n - 90^\circ \), (4) \( n - 180^\circ \). 

8. If the proportion, \( \frac{7}{9} = \frac{5}{m} \), has been rearranged to read \( \frac{7}{5} = \frac{9}{m} \), it has been rearranged by (1) inversion, (2) substitution, (3) subtraction, (4) alternation.

9. The sum of all the angles formed on one side of a line at a given point on the line is (1) a straight angle, (2) an acute angle, (3) a right angle, (4) 360 degrees.

10. An angle that is nine times as large as its complement is (1) \( 9^\circ \), (2) \( 162^\circ \), (3) \( 80^\circ \), (4) \( 81^\circ \).

11. The locus of the mid-points of parallel chords in a circle is (1) a circle, (2) tangents at end points, (3) perpendicular bisector of one of the chords, (4) the center of the circle.

12. The altitude to the hypotenuse of a right triangle divides the triangle into two triangles that are (1) isosceles, (2) congruent, (3) similar, (4) equilateral.

13. If two lines in a plane are perpendicular to the same line, they are (1) parallel, (2) equal, (3) bisected, (4) perpendicular to each other.

14. The areas of two similar triangles have the same ratio as (1) their altitudes, (2) their corresponding sides, (3) cubes of their corresponding sides, (4) the squares of their corresponding sides.

15. If two regular polygons have the same number of sides they are (1) similar, (2) equivalent, (3) congruent, (4) pentagons.

16. Two triangles having an angle of one equal to an angle of the other and the including sides proportional are (1) congruent, (2) equilateral, (3) similar, (4) equal.

17. The number of points a given distance from two intersecting lines is (1) two, (2) four, (3) one, (4) any number.

18. A central angle in a circle is formed by (1) two secants, (2) two chords, (3) a diameter and a tangent, (4) two radii.

19. If two angles of one triangle are equal respectively to two angles of another triangle, the triangles are (1) similar, (2) congruent, (3) equilateral, (4) equal.

20. The number of lines required in drawing the medians, altitudes, and angle bisectors in an isosceles triangle is (1) five, (2) six, (3) seven, (4) nine.
21. The median of a triangle divides the triangle into two triangles that are (a) isosceles, (b) equal, (c) congruent, (d) equilateral. .................. 21

22. The locus of all points a given distance from a given line segment is (a) two parallel lines, (b) two semicircles, (c) two circles, (d) two parallel line segments and two semicircles. ......................... 22

23. The acute angle formed by the hands of the clock at 2:00 P.M. is (a) 30°, (b) 45°, (c) 60°, (d) 90° .... 23

24. The supplement of an angle of 86 degrees is (a) 4°, (b) 94°, (c) 104°, (d) 184°. 24

25. The sum of the interior angles of a polygon equals (a) two straight angles, (b) two right angles, (c) (n - 2) right angles, (d) (n - 2) straight angles. 25

26. If a proportion \(\frac{m}{n} = \frac{v}{w}\), has been rearranged to read \(\frac{w}{v} = \frac{n}{m}\), it has been rearranged by (a) substitution, (b) inversion, (c) alternation, (d) subtraction. 26

27. A polygon with six equal sides is called a (a) decagon, (b) octagon, (c) pentagon, (d) hexagon. 27

28. An inscribed angle is formed by (a) two radii, (b) two chords intersecting within the circle, (c) two secants intersecting outside the circle, (d) two chords intersecting on the circumference. 28

29. One third of a straight angle is (a) 90 degrees, (b) 120 degrees, (c) 30 degrees, (d) 60 degrees. 29

30. The locus of the hub of a wheel turning on a level straight-away is (a) two lines, (b) a circle, (c) a straight line, (d) several circles. 30

31. The number of triangles that can be constructed on a fixed base, MN, with an altitude of six inches is (a) one, (b) two, (c) none, (d) an indefinite number. 31

32. The area of the largest triangle that can be inscribed in a semicircle whose radius is \(r\) is (a) \(r^2\), (b) \(r^3\), (c) \(2r^2\), (d) \(2r^3\). 32

33. Lines are concurrent if they (a) are everywhere equally distant, (b) intersect in one point, (c) lie in the same plane, (d) form a triangle. 33

34. When the shadow of a flag pole is longer than the pole itself, the sun's rays strike the earth at an angle of (a) less than 45°, (b) 45°, (c) more than 45°, (d) 90°. 34

35. A circle may be inscribed in any (a) triangle, (b) quadrilateral, (c) pentagon, (d) hexagon. 35
Directions: Solve the following exercises and write your answers on the proper spaces to the right. Answers involving radicals should be left in rationalized form. Use the space below each exercise for computation and drawings if you so desire.

1. If two angles are complementary and one is 10° larger than the other, how large is each angle? ( points) .. ___ 1

2. If the acute angles of a right triangle are each 45° and one side including the right angle is b, what is the hypotenuse in terms of the side b? ( points) ... ___ 2

3. A secant and a tangent drawn to a circle from an external point form an angle of 20°. If one arc intercepted on the circle is 70°, how large is the other arc? ( points) ......................... ___ 3

4. If the vertex angle of an isosceles triangle is three times as large as the base angle, how many degrees in each angle of the triangle? ( points) .................. ___ 4

5. How far from the center of a circle with a radius of 13 inches is a chord 24 inches long? ( points)..... ___ 5

6. The bases of a trapezoid are 16 and 27 inches, respectively. How long is the line joining the midpoints of the non-parallel sides? ( points)..... ___ 6

7. The sides of a triangle are 10, 12, and 16 inches, respectively. How long is the altitude to the longer side? ( points) ......................... ___ 7
8. If one side of a square is $s$, find the length of the diagonal in terms of $s$. (points)........... 8

9. The base of a triangle is 20 inches. How long is the segment joining the mid-points of the other two sides? (points)................................. 9

10. If the circumference of a circle is increased by 14 inches, by how much is the radius increased? Express the answer in terms of $\pi$. (points).... 10

11. One angle of a right triangle is 30°. How long is the hypotenuse in terms of the shorter leg, $b$, of the triangle? (points)......................... 11

12. What is the area of a square inscribed in a circle whose radius is 6 inches? (points)............. 12

13. If the altitude of an equilateral triangle is 18 inches, how long is the side of the triangle? (points)................................. 13

14. One angle of a rhombus is 120°. If the shorter diagonal is 12 inches, find the side of the rhombus? (points)................................. 14

15. One angle of a triangle is twice the second and the third is five times the second. How large is each angle? (points)................................. 15
PART III (total points ______)

CONSTRUCTION AND LOCUS PROBLEMS

Directions: You are to use straight edge and compasses in making the following constructions. Show all arcs, lines and points needed in the drawings. Letter your constructions very plainly. Do not describe the constructions unless asked to do so.

1. Locate a point which is equidistant from the three points, A, R, and S. (10 points)

A.

. R

. S

2. Construct the locus of vertex A of a right triangle having BC as hypotenuse. (10 points)

B __________ C

3. Divide the right angle AMR into three equal parts. (10 points)

A

M __________ R
4. Construct the locus of the vertices of triangles above $AB$ having $AB$ as the base and area equal to the area of $ABM$. (10 points)

5. Construct two circles tangent to the circle $P$ internally and with radii equal to $r$ and $r'$. (10 points)

6. Construct the locus of the midpoints of all chords, in circle $P$, equal to $XY$. (15 points)

Describe the steps taken in making this construction.
7. Construct a circle, outside of triangle ANG, and tangent to the
three lines AR, AL, and NG. ( points)

8. Construct a line that shall be tangent to circle C and perpendicular
to line l. ( points)

9. Construct the locus of points at distance d from the circle whose
center is M. ( points)
Directions: In each of the following problems you are to make a proof. Your proof must be clear, concise and complete to be acceptable. Number each statement and the corresponding reason. The blanks are to aid you in writing, but the number of blanks does not indicate the number of steps in the proof.

1. An angle inscribed in a circle is equal in degrees to half the intercepted arc. (points)

Given: 

To prove: 

Construct: 

Proof Statements
1. 

2. 

3. 

Reasons
1. 

2. 

3. 

---

---
2. Prove that any median of a triangle is less than half the perimeter. (points)

Given __________________________

To prove __________________________

Proof Statements Reasons
1. __________________________ 1. __________________________
   __________________________
   __________________________
   __________________________
   __________________________

5. If two medians of a triangle are equal the triangle is isosceles. (points)

Given triangle ABC with BX and CY medians to AC and AB respectively.

To prove that triangle ABC is isosceles.

State the steps you would use for an analytic proof.

Proof
1. ABC is isosceles if I can prove AB = AC, since an isosceles triangle has two equal sides by definition.

2. __________________________
   __________________________
   __________________________
   __________________________
   __________________________
   __________________________
4. Two lines perpendicular to the same line are parallel.

Given ________________
To prove ________________

Use the indirect method of proof.
Number your statements and reasons.

Proof Statements
1. Assume that AB and CD are not parallel. ____________________________
2. ____________________________ ____________________________
   ____________________________ ____________________________
   ____________________________ ____________________________
   ____________________________ ____________________________

5. The bisectors of the four angles of a rectangle form a square.

Given ____________________________
To prove ____________________________

Proof Statements
______________________________
______________________________
______________________________
______________________________
______________________________
______________________________
6. The area of a trapezoid is equal to half the product of the altitude and the sum of the bases. (points)

Given

To prove

Proof Statements Reasons

\[
\begin{align*}
\text{Given:} & \quad \text{M, N, O, P, R, S are points on the line segment.} \\
\text{To prove:} & \quad \text{Area of trapezoid } = \frac{1}{2} \times \text{base} \times \text{height} \\
\text{Proof:} & \quad \text{Statements} \\
\text{Reasons} \\
\end{align*}
\]
C. Direction for Scoring and Keys

Directions for Scoring

The Sectional Plane Geometry Test

1934

Part I----Each item in Part I should be scored 3 points when correctly solved. While the directions on the examination called for underlining the correct word or phrase in addition to writing the correct "letter" on the blank at the right, only the "letter" scores should be considered.

Part II---Each correct answer in Part II has been assigned an individual possible score. In cases where there are two or more answers, the points for that exercise should be divided equally among the parts.

Part III--Each item or exercise in Part III has been assigned an individual possible score. In Exercise 6 construction and description should each count 10 points. In Exercise 8 only one tangent is required for full credit.

Part IV---The most difficult part to score is Part IV.

Each proof has been assigned 30 points. It is essential in Exercises 3 and 4 of Part IV that the types of proof required be submitted if credit is to be granted. The purpose of the examination is to measure ability to use various methods of proof. Partial credit in the solution of each exercise
should be given for each correct statement when accompanied by a sound reason. The following suggestions are made concerning partial credit for an unfinished proof.

(1) Allow not more than 25 points credit for any incomplete proof.

(2) Allow 3 points for each correct statement. To illustrate in Exercise 1, allow
3 points for correct "Given"
3 points for correct "To prove"
3 points for correct "Construct"
3 points for each correct step in the proof when accompanied by a sound reason.

(3) An exception to the above should be made in Exercise 3 in which analysis is required. Here 35 points is allowed for a complete analysis. Allow 25 points for an unfinished proof in case the pupil gives unquestionable evidence that he understands what an analytic proof is. For anything less than evidence of understanding this type of proof certainly no credit should be allowed.
PART II (total points 12)

**CONSTRUCT Key for Parts I and II**

Directions: You are to use straight edge and compass in the following constructions. Show all arcs. Better your constructions very carefully. Describe the constructions unless asked to do otherwise.

**Part I**

1. Locate a point which is equidistant from A and B.
2. 4.15
3. 6. 12
4. 9. 1.5
5. 10. 14
6. 12. 72
7. 14. 12
8. 15. 22.5, 45, 112.5

**Part II**

1. $40.50$
2. $b\sqrt{2}$
3. 30, 110
4. 36, 56, 108
5. 5
6. 21.5
7. 7.5
8. $5\sqrt{2}$
9. 10
10. 14/24
11. 2b
12. 72
13. $12\sqrt{3}$
14. 12
15. 22.5, 45, 112.5

2. Construct the length of vertex A of a right triangle using as hypotenuse (10 points)

3. Divide the right angle ABD into three equal parts.
PART II (total points 120)

CONSTRUCTION AND LOCUS PROBLEMS

Directions: You are to use straight edge and compasses in making the following constructions. Show all arcs, lines and points needed in the drawings. Letter your constructions very plainly. Do not describe the constructions unless asked to do so.

1. Locate a point which is equidistant from the three points, A, R, and S. (10 points)

2. Construct the locus of vertex A of a right triangle having BC as hypotenuse. (10 points)

3. Divide the right angle AMR into three equal parts. (10 points)
4. Construct the locus of the vertices of triangles above AB having AB as the base and area equal to the area of ABM. (10 points)

5. Construct two circles tangent internally and with radii equal to r and r'. (10 points)

6. Construct the locus of the midpoints of all chords, in circle P, equal to XY. (15 points)

Describe the steps taken in making this construction.

1. Draw a chord equal to XY.
2. Drop a perpendicular from center of circle to XY, meeting XY in point O.
3. Draw a circle with center P and radius PO.
7. Construct a circle, outside of triangle \( \triangle \text{ANG} \), and tangent to the three lines \( \text{AR}, \text{AL}, \) and \( \text{NG} \). (20 points)

8. Construct a line that shall be tangent to circle \( C \) and perpendicular to line \( l \). (20 points)

9. Construct the locus of points at distance \( d \) from the circle whose center is \( M \). (15 points)
Directions: In each of the following problems you are to make a proof. Your proof must be clear, concise and complete to be acceptable. Number each statement and the corresponding reason. The blanks are to aid you in writing, but the number of blanks does not indicate the number of steps in the proof.

1. An angle inscribed in a circle is equal in degrees to half the intercepted arc. (35 points)

Given Circle O with inscribed angle APB
To prove \( \angle APB \) is measured by
\[ \frac{1}{2} \text{arc AB} \]
Construct Diameter PM and radii OA and OB

Proof Statements
1. OB=OP
2. \( \triangle OBP \) is isosceles
3. \( \angle MOB = 2 \angle OBP \)
4. \( \angle AOM = 2 \angle APO \)
5. \( \angle AOB = 2 \angle APB \)
6. \( \angle AOB \) is measured by arc AB
7. \( 2 \angle APB \) is measured by arc AB
8. \( \angle APB \) is measured by \( \frac{1}{2} \) arc AB

Reasons
1. Radii of same circle are equal
2. If two sides of a triangle are equal, the triangle is isosceles
3. The exterior angle at the vertex of an isosceles is twice either base angle.
4. By repetition of steps 1 to 3.
5. If equals are added to equals the sums are equal.
6. A central angle is measured by its intercepted arc.
7. Equals substituted for equals.
8. Division axiom.
2. Prove that any median of a triangle is less than half the perimeter.

(30 points)

Given \( \triangle ABC \) with median \( m \) to side \( AB \)

To prove \( m < \frac{1}{2} (a + b + c) \)

Proof

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( m &lt; b + \frac{1}{2}c )</td>
<td>1. The sum of two sides of a triangle is greater than the third.</td>
</tr>
<tr>
<td>2. ( m &lt; a + \frac{1}{2}c )</td>
<td>2. Same as (1).</td>
</tr>
<tr>
<td>3. ( 2m &lt; a + b + c )</td>
<td>3. If unequals are added to unequals in the same order, the sums are unequal in the same order.</td>
</tr>
<tr>
<td>4. ( m &lt; \frac{1}{2} (a + b + c) )</td>
<td>4. If unequals are divided by positive equals, the results are unequal in the same order.</td>
</tr>
</tbody>
</table>

3. If two medians of a triangle are equal the triangle is isosceles.

(35 points)

Given triangle \( \triangle ABC \) with \( \text{BX} \) and \( \text{CY} \) medians to \( AC \) and \( AB \) respectively.

To prove that triangle \( \triangle ABC \) is isosceles.

State the steps you would use for an analytic proof.

Proof

1. \( \triangle ABC \) is isosceles if I can prove \( AB = AC \), since an isosceles triangle has two equal sides by definition.

2. \( AB = AC \) if I can prove \( \angle C = \angle B \), since if two angles of a triangle are equal the sides opposite are equal.

3. \( \angle C = \angle B \) if \( \triangle XCB \cong \triangle YCB \), since corresponding angles of congruent triangles are equal.

4. \( \triangle XCB \cong \triangle YCB \) if \( \angle MCB < \angle MBC \) \( \angle CB = \angle CB \) (Identical) \( \text{XB} = \text{YC} \) (Given)

5. \( \angle MCB < \angle MBC \) if \( \triangle CMB \) is isosceles

6. \( \triangle CMB \) is isosceles since \( \text{CY} = \text{XB} \), \( 2/3 \text{CY} = 2/3 \text{XB} \) and \( \text{CM} = \text{MB} \)
4. Two lines perpendicular to the same line are parallel.
Given CD \perp XY and AB \perp XY
To prove CD \parallel AB
Use the indirect method of proof. Number your statements and reasons.
Proof Statements
1. Assume that AB and CD are not parallel.
2. Then AB and CD will meet as some point P
3. From point P we now have two lines perpendicular to XY
4. This is impossible and \therefore CD and AB are parallel.
5. The bisectors of the four angles of a rectangle form a square.

Given Rectangle EF GH with angle bisectors EC, FA, GB, HC
To prove DCBA is a square
Proof Statements
1. \angle m = \angle z = 45^\circ
2. \angle FAG + m + z = 180^\circ
3. \angle FAG = 180^\circ - (m + z)
4. \angle FAG = 90^\circ
5. \angle ADC = \angle ABC = 90^\circ
6. \angle ADCB is a rectangle
7. \angle FAG \cong \angle ECH
   \begin{align*}
   (1) & \quad EH = FG \\
   (2) & \quad \angle t = \angle z = 45^\circ \\
   (3) & \quad \angle w = \angle m = 45^\circ \\
   \end{align*}
8. $\overline{AF} = \overline{EC}$

9. $\angle x = \angle y = 45^\circ$

10. $\overline{FD} = \overline{ED}$

8. Corresponding sides of congruent triangles.

9. Each is half a right angle.

10. If two angles of a triangle are equal, the sides opposite are equal.

11. Subtraction axiom.

12. From figure.

13. $\overline{AD} = \overline{DC}$

14. If a rectangle has two adjacent sides equal, it is a square.
6. The area of a trapezoid is equal to half the product of the altitude and the sum of the bases. (30 points)

Given Trapezoid TSRM with bases $b'$ and $b$, and altitude $h$

To prove that trapezoid $TSRM = \frac{1}{2} (b + b')$

Proof

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construct diagonal MS</td>
<td></td>
</tr>
<tr>
<td>1. Area of $\triangle MTS = \frac{1}{2} hb'$</td>
<td>1. The area of a triangle is half the product of its base and height.</td>
</tr>
<tr>
<td>2. Area of $\triangle MRS = \frac{1}{2} hb$</td>
<td>2. Same as (1)</td>
</tr>
<tr>
<td>3. Area of TSMR = $\frac{1}{2} hb + \frac{1}{2} hb'$</td>
<td>3. If equals are added to equals the sums are equal.</td>
</tr>
<tr>
<td>4. Area of TSMR = $\frac{1}{2} h (b + b')$</td>
<td>4. Because $\frac{1}{2} hb + \frac{1}{2} hb'$ can be written in factored form.</td>
</tr>
</tbody>
</table>