A STUDY OF THE POLAR CHART PROBLEMS OF GLOBAL NAVIGATION

A Thesis
Presented to
the Faculty of the Graduate Council
Indiana State Teachers College

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts in Education

by
Winston P. Wythe
1946
The thesis of Winston P. Wythe.

Contribution of the Graduate School, Indiana State Teachers College, Number 585, under the title

A Study of The Polar Chart Problems of

Global Navigation

is hereby approved as counting toward the completion of the Master's degree in the amount of 4 hours' credit.

Committee on thesis:

[Signatures]

Representative of the English Department:

[Signature]

Date of Acceptance, September 12, 1946
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>General Statement</td>
<td>1</td>
</tr>
<tr>
<td>The Problem</td>
<td>1</td>
</tr>
<tr>
<td>Justification of the Topic</td>
<td>2</td>
</tr>
<tr>
<td>Scope and Limitations</td>
<td>3</td>
</tr>
<tr>
<td>Organization of Chapters</td>
<td>4</td>
</tr>
<tr>
<td>Survey of Previous Research</td>
<td>4</td>
</tr>
<tr>
<td>II. THIS IS NOT A REVIEW OF NAVIGATION</td>
<td>6</td>
</tr>
<tr>
<td>III. THE POLAR CHART</td>
<td>8</td>
</tr>
<tr>
<td>An explanation of the projection of a spherical surface on a polar cotangent plane</td>
<td>8</td>
</tr>
<tr>
<td>IV. GETTING ACQUAINTED WITH POLAR CHARTS</td>
<td>14</td>
</tr>
<tr>
<td>Elliptical Sight with Known DRLo</td>
<td>15</td>
</tr>
<tr>
<td>Elliptical Sight with Known DRLat</td>
<td>20</td>
</tr>
<tr>
<td>The Problem of Great Circle Sailing</td>
<td>22</td>
</tr>
<tr>
<td>V. IF THE SHIP IS REALLY LOST</td>
<td>26</td>
</tr>
<tr>
<td>The Bi-Circular (Two-Azimuth) Sight</td>
<td>26</td>
</tr>
<tr>
<td>The Elliptical Mid-Morning Sight</td>
<td>31</td>
</tr>
<tr>
<td>The Parabolic Mid-Morning Sight</td>
<td>35</td>
</tr>
<tr>
<td>The Hyperbolic Mid-Morning Sight</td>
<td>38</td>
</tr>
<tr>
<td>The Hyperbolic Mid-Morning Sight (Hemisphere not Known)</td>
<td>41</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>44</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The General Polar Plane Projection</td>
<td>10</td>
</tr>
<tr>
<td>2. The Elliptical Projection of the Circle of Position</td>
<td>11</td>
</tr>
<tr>
<td>3. The Parabolic Projection of the Circle of Position</td>
<td>12</td>
</tr>
<tr>
<td>4. The Hyperbolic Projection of the Circle of Position</td>
<td>13</td>
</tr>
<tr>
<td>5. Elliptical Projection Showing, Independently, a Known or Assumed DRLo and DRLat</td>
<td>17</td>
</tr>
<tr>
<td>6. Great Circle Course Projected to the Polar Plane</td>
<td>23</td>
</tr>
<tr>
<td>7. Graphical Representation of the Bi-Circular (Two-Azimuth) Sight</td>
<td>28</td>
</tr>
<tr>
<td>8. The Elliptical Mid-Morning Projection</td>
<td>33</td>
</tr>
<tr>
<td>9. The Parabolic Mid-Morning Projection</td>
<td>36</td>
</tr>
<tr>
<td>10. The Hyperbolic Mid-Morning Projection</td>
<td>39</td>
</tr>
<tr>
<td>11. Reversal of the Projection for the Hyperbolic Mid-Morning Sight</td>
<td>42</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

I. GENERAL STATEMENT

The author's first concern with the polar chart problems of navigation began on 21 March 1931 at Adler Planetarium. It was there that the lecturer, from the faculty of Northwestern University, stated that in order to navigate by the celestial bodies, the navigator must have a reasonable dead reckoning point and an accurate means of determining local time. It was disappointing to learn that in the colossal grandeur of a night at sea the navigator could only confirm what he already pretty well knew. Previous thought on the problem did not crystallize into action until World War II when many times the burden of successful navigation was put upon the author.

II. THE PROBLEM

It has become evident that emergency situations arise frequently and from various causes in which the dead reckoning point is not reasonable. In that case, any local time figure has little value, and the problem of establishing position becomes one of a long series of problems that may be solved by polar chart methods. At present, there seems no other way, with the possible exception of the work of Captain Sumner, which the writer has been unable to review.
Since DR and local time are considered valueless in such emergencies, they do not become a part of the problem, as in other recognized systems, all widely used on the sea. So, assuming an emergency, this work enters the problem with Greenwich time, direction, altitude, azimuth, and the usual corrections for index, height of eye, refraction, equation of time, and R. A., already made before entering into the problem with any argument whatever. Since this work is in no sense elementary navigation and because it requires an experienced navigator to apply it, there should be no objection to their omission in the interest of simplifying the method.

III. JUSTIFICATION OF THE TOPIC

It is not meant by the author of this monograph that the material here contained will supplant any of the common systems of navigation. All too often, however, even in modern times, ships are blown far off course by wind or set, lose the accuracy of dead reckoning during a running battle, or get lost because of faulty navigation such as continuing on one course too long, ending on forgotten beaches, unfrequented islands, or in the polar seas. In the latter, the use of this method is particularly of value since its application is very apt in the high latitudes.

Accidents of many kinds happen at sea in spite of the eternal vigilance of the watch, and many of them happen to
radio directional equipment, radar, and Loran navigational gear. When these mechanical means become inoperative from any cause, the fate of the ship and all aboard depends on the fundamental knowledge coming from the navigator's bridge.

IV. SCOPE AND LIMITATIONS

The series of problems here recorded does not include all the problems of the polar chart system for three reasons: (1) Mathematicians aboard ship can establish position in an emergency with the methods here presented, (2) Further presentation of the series would make this a study of academic interest to mathematicians only, and, (3) Another system for using polar charts is already in limited use.

It is to be hoped that a convenient routine for navigation by polar charts will soon be developed because it makes possible the navigation of a whole hemisphere at one time, eliminating the clumsy Mercator chart. A clever mathematician could well use one blank chart for navigating the entire earth, as is hinted in the final problem, but expansion into the subject, as is also the case of the general sights, is beyond the scope of this work.

The terms used herein have the standard definitions of astronomy, mathematics, and navigation. Because it is futile to attempt to lead the elementary reader by the hand, and because the terms used are all familiar to the initiated, a listing of the definitions would be superfluous.
V. ORGANIZATION OF CHAPTERS

The Introduction points out the problem of establishing position when the only arguments available are those immediately at hand.

Chapter II is meant to acquaint the reader with the tools necessary for an emergency job. It is to be regretted that no way has been found to communicate to the navigator the means for preventing excitement, alarm, and confusion.

Chapter III is a concise definition and exposition of the projection of a spherical surface to the polar plane.

Chapter IV deals with two elementary problems in the use of Cartesian coordinates on the polar plane. The subject of great circle sailing is also taken up because it helps to develop facility in the system and at the same time throws considerable light on a confusing problem.

Chapter V investigates four procedures any one of which a navigator may follow if an emergency arises and the ship is really lost.

VI. SURVEY OF PREVIOUS RESEARCH

It seems unlikely that the able astronomers, navigators, and mathematicians of the past would have overlooked the solution of this long series of problems. The author was unable, however, to find any such work in Widener Library at Harvard University, or at Boston University, or in Boston
Public Library. In the latter, was found a meager reference to an early work of Huygens that might have led to the solutions, but that early work could not be found there nor at Widener Library.

Much of the later work of this contribution was done aboard various ships of the U. S. Navy during otherwise idle hours at sea. The writer wishes especially to thank Dr. Robert Bruce, Head of the Department of Mathematics, College of Liberal Arts, Boston University, for permission to use the library with full freedom.
CHAPTER II

THIS IS NOT A REVIEW OF NAVIGATION

There is certainly no reason to review any of the works in navigation for they have already been well done. Every seaman knows that the compasses should be compensated and calibrated before leaving port: The best set of tables extant for that purpose is to be found in H. O. 71 and the best and shortest text on compass (magnetic) compensation is H. O. 226, written by Spencer and Kucera. The gyro-compass requires an expert with much training. The seaman may become adept on a magnetic compass in two months. Whatever Nautical Almanac is current is also essential, as are the sextant and an accurate azimuth circle. Some ability to use coordinate mathematics and a good table of tangents are also required in this work.

In other words, the tools are already to be found in stores that serve seamen; in bookstores, in the Hydrographic Office, or the Naval Observatory. Courses in the common methods of navigation are always available to seamen. This method is simply a way of using the tools when DR and local time are afloul. That is when the ship is lost. Fortunately, one needs only to know Greenwich time and North or South. Compasses are not likely to lead the ship astray unless they have been tampered with; so they can usually be relied upon. If not, the navigator may as well stop and find North by the
method of highest altitude, since the course is assumed wrong. A navigator can determine his position if he does that. But suppose he cannot get a "noon" sight, or suppose he can determine latitude only from it. Then he must use other means. The sky is full of stars.
CHAPTER III

THE POLAR CHART

A polar chart is defined as a plane through either pole and is parallel to the equator. That places the pole in the center of the chart. The latitude circles circumscribe the pole at distances corresponding to the cotangent of the latitude. Thus, a polar chart is a sheet showing the pole in the center of many concentric circles, spaced at ever increasing distances from each other, until finally the circle of 0 latitude is an infinite distance away and cannot be shown on any chart. Thus only one hemisphere can be shown at a time, but that fact is unimportant as navigation along the equator presents few problems. The question arises: How can a navigator determine which hemisphere he is in if he is not on the equator? He need know the heavens, his instruments must be accurate, and he must use them well. Training in these requirements is also outside the scope of this work. A clear mental picture of the chart is necessary.

Every navigator knows that when a sight is taken with the sextant, he establishes a line of position, or Sumner line, on his Mercator chart. If this line is shown on a sphere as illustrated on the Plate immediately following, it becomes a circle of position. But circles on spherical surfaces are hard to deal with; so we project the circle of position onto the polar plane where it becomes another member
of the conic sections. It is always an ellipse, a parabola, or an hyperbola, depending on whether it does not reach the equator, reaches it, or reaches below it, respectively.

The next four pages show the projections.
CHAPTER IV

GETTING ACQUAINTED WITH POLAR CHARTS

Mathematicians may soon improve on the methods of using polar coordinate charts, but the basic principles are never going to change. For that reason, several simpler problems will be discussed in order to develop facility in their use. All the problems are stripped of all except essentials. The problems discussed are as follows:

1. The determination of position when only DRLo is known.

2. The determination of position when only DRLat is known.

3. The problem of great circle sailing.

In the end, this work will have taken the reader through the right angle sights. The problems of more complex sights are for mathematicians of higher attainments than are usually found aboard ships.

There will be the criticism that the calculating requires some time and that there has been little or no effort to reduce it to a handy routine. Neither is losing one's way routine, but when it happens the navigator has all eternity in which to extricate his ship and his men.
I. ELLIPTICAL SIGHT WITH A KNOWN DRLo

The problems used here are not very likely to exist in practical navigation, but they may. They are used chiefly to acquaint the reader with the main problem of being forced to establish position when only Greenwich time and direction are known.

In the chart, note that the pole is at N. For convenience, suppose it is the North pole. Every line that may be drawn through it extends south toward the equator; which cannot be shown, but even a small chart reaches within five degrees of it. Likewise, note that what would be the circle of position on a sphere has become an ellipse on the polar plane. The example of the ellipse was chosen because it facilitates drawing and yet contains all the problems in the analytics of conic sections.

Note that GP is the geographical position of the celestial body and is placed on the X axis at cot Decl. If the navigator has made careful correction and has properly corrected for the equation of time and the R. A., where applicable, the GP will be correct, since for the purposes of these problems its meridian is zero. GP is one focus of the ellipse. The ellipse crosses the X axis at two points representing the upper and lower limits of the circle of position on the earth's surface, which points are determined by the cotangent of the maximum and minimum latitudes. From
these, the other focus and the center \((h,k)\) can be found. The semimajor and semiminor axes can be determined by the remaining steps in analysis.

Next, since the DRLo is correct, draw a line through \(N\) representing the angular distance away from the meridian of the GP.

---

Now the equations of the ellipse of position and of the line representing DRLo in Cartesian coordinates are synthesized and solved simultaneously. The two reasonable roots will give the coordinate position relative to the GP. The latitude may be found from the equation,

$$\cot \text{Lat.} = \sqrt{\frac{x^2 + y^2}{b^2}}$$

An example follows:

Alt. 65° Decl. 32° DRLo 26° 34' W of GP

GP = geographical position of body plotted on horizontal axis,

$$b = \cot \text{lower intercept} - \cot \text{upper intercept}$$

$$d = b - (\cot \text{Lat. GP} - \cot \text{upper intercept})$$

$$a = \sqrt{b^2 - d^2}$$

SOLUTION

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Ellipse of position (equal alt.) when it does not extend to the equator.

$$x \pm \tan \text{DRLo} x y = 0$$

Meridian of DRLo with respect GP.

Substitute the correct values in the two preceding equations and solve simultaneously for the reasonable roots.

\[\text{The experienced navigator will remember that there is always a correction for latitude. By the method used here the correction involves the use of derivatives. The magnitude is relatively unimportant and will be ignored in this work, in the interest of simplifying the method.}\]
The coordinate construction of the problem is shown in Fig. 5.

Substituting in the foregoing type equations gives the pair:

\[ x \pm 1.99984y = 0 \]
\[ \frac{(x - 4.396827)^2}{(5.7474194)^2} + \frac{y^2}{(2.4646112)^2} = 1 \]

Solve the linear equation for \( x \) and substitute that value in the quadratic, the result is:

\[ 1.5642458x^2 - 8.7937654x + 5.28893567 = 0 \]
\[ x = + .684894 \]
\[ y = - .3424743 \]

Obviously the construction of Fig. 5 shows this pair of roots to be the reasonable solution.

Substituting in

\[ \cot \text{ Lat.} = \sqrt{\frac{x^2 + y^2}{x^2 + y^2}} \]
\[ = \sqrt{.58636843} \]
\[ = .765093 \]

\[ \therefore \text{Lat.} = 52 \text{ deg. 34 min. 49 sec. North.} \]
II. ELLIPTICAL SIGHT WITH A KNOWN DRLat.

The method used here is the same as in the preceding problem. For the DRLat, a circle is circumscribed about the pole with a radius equal to cot Lat. Then the equations are built up for the ellipse of position and the circle of DRLat and they are solved simultaneously. This problem, like all other problems in this work, is solved in Cartesian coordinates and gives the coordinate position relative to the pole and the GP. The transposition to the latitude and longitude system is then made.

When the two equations are solved simultaneously, longitude with respect to the GP can be found,

\[ \tan Lo = x \]

The equation of the ellipse is built up in the same way as before. The Cartesian equation of the circle of DRLat. is

\[ x^2 + y^2 = \cot^2 \text{DRLat}. \]

In the following example,

Alt. 65°  Decl. 32°N.  DRLat. 45°N.

With the correct use of the type equations for the ellipse of position and the circle representing DRLat, the pair is:

\[ x^2 + y^2 = 1 \]

\[ \frac{(x - 4.396827)^2}{(3.7474154)^2} + \frac{y^2}{(2.4646112)^2} = 1 \]

The pair is accurately represented in Fig. 5.
Solving the first equation for $y$ and substituting the value in the second results in:

$$1.2566222x^2 + 6.793654x - 7.545891 = 0$$

The following reasonable pair of roots will be found in the solution:

$$x = + .7728042$$

$$y = - .63464$$

Substituting these values in

$$\tan Lo = \frac{y}{x}$$

$$\tan Lo = - .821217$$

$$Lo = 39 \text{ deg.} \ 23 \text{ min.} \ 36 \text{ sec.} \ \text{West of GP}$$
III. THE PROBLEM OF GREAT CIRCLE SAILING

For a great many years this problem has been more confusing than enlightening to most seamen. On a polar chart it becomes one of the simplest in navigation. All that is necessary is to plot accurately the point of departure and the point of destination and draw a line through them. If there are no obstructions, the course can be sailed.\(^3\)

The procedure is to plot the points in Cartesian coordinates on a polar cotangent chart and draw the line through them. Then a line is drawn from the point of departure through the pole and the equation for each line synthesized. By the method of treatment shown following this discussion, the required initial course can be found in a few minutes. Likewise the required course can be found for any degree of longitude along the course. Also the maximum latitude reached by the course can be determined. Sailor, beware if maximum latitude is not reached at the right time. An entire great circle course thousands of miles long can be precalculated and sailed by dead reckoning. The calculations require a little more than three hours. Finding the required initial course from any point on the globe to any point on the globe is not at all difficult. If both are in the same hemisphere, it is quite easy.

---

\(^3\) The mathematical proof that a great circle course projected on a polar plane is a straight line has long been known and is outside the scope of this work.
From a study of Fig. 6 we see that M₂ represents the point of departure, N the destination, and M₁ represents the point of highest latitude reached along a great circle course between M₂ and N. The notations X and Y are used to define them in coordinates. OG represents Greenwich meridian and in this case coincides with the Y axis, but not necessarily so. However, the points of departure and of destination are always plotted with reference to OG, wherever it lies.

Problem 1 of great circle sailing:

To determine the great circle course from any position M₂ when the position of the destination N is known.

Lay out fix M₂ as X₂ Y₂ and N as XY on a polar chart where cot 45 deg. equals unity. This gives the positions in rectangular coordinates.

\[ OM₂ = \cot \text{Lat.} \]
\[ X₂ = \cos (90 - Lo₂) \times OM₂ \]
\[ Y₂ = \sin (90 - Lo₂) \times OM₂ \]
\[ ON = \cot \text{Lat.} \quad (\text{of } N) \]
\[ X = \cos (90 - Lo) \times ON \]
\[ Y = \sin (90 - Lo) \times ON \]
\[ m₁ = \frac{Y₂ - Y}{X₂ - X} \]
\[ m₂ = \tan (90 - Lo₂) \]
\[ \tan B = \frac{m₁ - m₂}{1 + m₁m₂} \]

angle B = required great circle course at M₂.
Problem 2 of great circle sailing:

To determine the maximum latitude reached on a given great circle course:

Construct $O\overline{M_1}$ (a meridian) perpendicular to the course line.

The equation of $M_2N$ is:

$$y - Y = \frac{x_2 - Y}{x_2 - X} (x - X)$$

The equation of $OM_1$ is:

$$y - \frac{(x_2 - X)}{(x_2 - Y)} x = 0$$

Solve for $x$ and $y$. Substituting $x = x_1$ and $y = y_1$ gives the position of $M_1$ in coordinates.

$OM_1$ is a meridian and may be established here:

$$\tan \text{ angle } M_1OX = \frac{Y_1}{X_1}$$

Angle $M_1OX$ may be taken from a table of tangents.

$Lo = 90^\circ - \text{angle } M_1OX$

To establish maximum latitude along the course:

$$OM_1 = \frac{Y_1}{\sin \text{ angle } M_1OX}$$

$$OM_1 = \cot. \text{ max. Lat.}$$

Max. Lat. may be taken from a table of cotangents.
CHAPTER V

IF THE SHIP IS REALLY LOST

I. THE BI-CIRCULAR (TWO-AZIMUTH) SIGHT

(Zn greater than 90°)

This problem seems not quite to belong in the same family with the others, but it is a near relative. It is recounted here because it is an easy way out if the ship is really lost. The sky is often clear enough to read azimuths on two celestial bodies as is required.

The navigator should generally follow the accepted methods of navigation which have been in use for many years and have been reduced to a convenient routine. The methods illustrated in this thesis are for use only when there seems no way out but to rely on data immediately at hand.

The method of this problem is based on the theorem that any angle inscribed in a circle is equal to one-half the intercepted arc. The projection of the circle of position has no part in the problem. The circles of equal azimuths are the solution.

Azimuth sights on two celestial bodies must be taken at the same time. Successive sights on the same body may be used if care is taken not to change position. The GP's are plotted in rectangular coordinates on a polar plane. If the GP's are for fixed stars their relative positions do not change. From this point, the problem is easy if ship and stars are all three in the same hemisphere. As this work is
concerned only with the illustration of principles, the case in which both azimuths are greater than 90° will be used.

When two circles are constructed, each passing through the pole and each passing through its respective GP, the center of each will lie on the perpendicular bisector of a line through the GP and the pole, and at a distance from it proportional to the tangent of the azimuth. The point of intersection P is the ship's position relative to GP₁.
Study Fig. 7 carefully. It will be seen that $OGP_1$ represents the meridian of the first celestial body $B_1$, and is at Decl. $30^\circ$ N. Placing it on one of the axes simplifies the problem. Since it lies on the observer's left, a circle passing through it and the pole will be offset on the perpendicular bisector of $OGP_1$ at a distance equal to $\frac{1}{2} OGP_1 \times \tan (Zn - 90^\circ)$. $C_1$ is therefore the center of the circle and is expressed in rectangular coordinates. $C_2$ is likewise the center of the second required circle and is also expressed in coordinates with reference to the pole at $O$. $B_2$ is at Decl. $20^\circ$ N. For convenience in calculation, the two azimuths are taken at $120^\circ$ East and West respectively from North.

With $h, k$ and $H, K$ representing the centers of the two circles their equations become

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - H)^2 + (y - K)^2 = R^2$$

With the correct values substituted, these become

$$(x + .4619768)^2 + (y - .80016725)^2 = .5536893844 \quad (r = .9239533)$$

$$(x - 1.4491177)^2 + (y - .64518882)^2 = 2.5162105 \quad (R = 1.586257)$$

Expanding these and subtracting one from the other evolves the linear equation:

$$3.822189x - .3099569y = 0$$

which is the equation for the meridian of position OP. From this

$$x = .081094y$$
Using this value in the equation for the circle the center of which is at $C_1$:

\[ y = 1.5154414 \text{ and } x = 1.22893205 \]

\[
\cot \text{ Lat.} = \sqrt{\frac{x^2 + y^2}{2.31166534}} = 1.520416
\]

Lat. = 33 deg. 20 min. 01 sec. N.

\[
\tan \text{ Lo} = x = \frac{1.22893205}{1.5154414}
\]

Lo = 4 deg. 38 min. 10.8 sec. W. of meridian of GP$_1$. 
II. THE ELLIPTICAL MID-MORNING SIGHT

In this problem the navigator has no accurate means of predicting the exact time when he will be able to get a mid-morning or mid-afternoon sight, because he does not know his position. He must be prepared to catch it when it comes. It requires only that the ship be in a greater latitude than the celestial body, and that they be in the same hemisphere. Any right angle sight depends on the existence of this condition. Since most dangers to navigation exist only in the high latitudes, the problem may often be extremely applicable. Like many problems in celestial navigation, the method seems to break down when the circle of position extends over the pole.

The construction of the ellipse remains the same, represented in rectangular coordinates on a polar chart.

When a circle is constructed through the GP and the pole, its center will lie on the meridian of the GP and exactly half way to the pole. The circle and the ellipse intersect at P and P'. One of them is the ship's position, depending on whether it was east or west of the body at the time of the sight. The distance away from the GP depends on the altitude as determined by the sextant, and that is what the ellipse represents. The simultaneous solution of the two curves represents the coordinate position with respect to the pole and the GP.
Notice that the angle $\overline{OPGP}$ intercepts $180^\circ$ on the circle and is therefore a right angle.

The problem is accurately represented in Fig. 8.
Example: Alt. 65° Zn 90° or 270° Decl. 32°.

The equations are synthesized in the following forms:

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \text{ for the ellipse, and}
\]

\[
(x - H)^2 + y^2 = R^2. \text{ When the azimuth makes a right angle with North, } H = \frac{OGP}{2}
\]

When the correct values are used in the above equations, they become

\[
\frac{(x - 4.396827)^2}{(3.7474194)^2} + \frac{y^2}{(2.4646112)^2} = 1 \text{ and}
\]

\[
(x - .800167)^2 + y^2 = .64027; \text{ the latter becoming}
\]

\[
y^2 = -x^2 + 1.600334 \text{ which is substituted in the equation for the ellipse.}
\]

After expansion and substitution it reduces to

\[-.0934188x^2 - .3637276x + .3766187 = 0 \text{ from which}
\]

\[
x = +.851543 \text{ and } y = -.798518
\]

\[
\cot \text{ Lat} = \frac{x^2 + y^2}{x^2 + y^2} = 1.1673715
\]

\[
\text{Lat} = 40 \text{ deg. } 35 \text{ min. } 3.1 \text{ sec. North.}
\]

\[
\tan \text{ Lo} = \frac{y}{x} = .95707
\]

\[
\text{Lo} = 43 \text{ deg. } 9 \text{ min. } 55.9 \text{ sec. West of meridian of GP.}
\]
III. THE PARABOLIC MID-MORNING SIGHT

The problem as it concerns the parabola of position is only for the rare cases when the circle of position on the globe would reach exactly to the equator. The only really justifiable reason for including it here is to provide continuity to the next case, where it reaches below the equator, and on a polar plane becomes an hyperbola.

The equations for the parabola and the circle are built up in the same way as before and again solved simultaneously. The ship's position relative to GP is either East or West of GP depending on whether it is East or West of the celestial body when the sight is taken.

Example: Alt. $67^\circ\ 30'$ Decl. $22^\circ\ 30'$ Azimuth $90^\circ$ or $270^\circ$

$OC = \frac{1}{2} OGP = R$ (radius of circle of equal azimuths).

$OV = \cot \text{Lat. of intercept on meridian of GP}$.

The two equations will be of the forms

$y^2 = 4a(x - h)$ for the parabola of position, and

$(x - H)^2 + y^2 = R^2$ for the circle of equal azimuths.

Substituting the correct values in the above equations

$y^2 = 4 \times 1.4142136(x - 1)$ for the parabola, and

$(x - 1.2071068)^2 + y^2 = 1.4571068$ for the circle of equal azimuths. Expanding and substituting, these become

$x^2 + 3.2426408x - 5.6568544 = 0$ from which

$x = \pm 1.25714$ and $y = \pm 1.2060695$. 
Returning to the equalities

\[ \cot \text{Lat} = \frac{1}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \tan \text{Lo} = \frac{y}{x} \]

\[ \cot \text{Lat} = 1.7421264. \]

\( \text{Lat.} = 29 \text{ deg. 51 min. 23 sec. North.} \)

\[ \tan \text{Lo} = .9533764 \]

\( \text{Lo} = 43 \text{ deg. 48 min. 44 sec. West of meridian of GP.} \)
IV. THE HYPERBOLIC MID-MORNING SIGHT

The hyperbolic mid-morning sight presents no unusual problems. It is necessary only to project the circle of position both forward and backward to the polar plane; then to build up the equations of the hyperbola and the circle and solve them simultaneously. The ship's position is at P or P', depending on whether it was east or west of the celestial body at the time of the sight.

Example: Alt. 60° Decl. 15° Zn 90° or 270°.

GP = geographical position of celestial body.

OD' = projection of lower intercept of circle of position on meridian of GP.

D = projection of upper intercept on meridian of GP.

D'GP' = DGP for construction purposes.

C' = midpoint between GP and GP'. (Intercept of asymptotes).

CGP = R (radius of circle of equal azimuths).

The correct equations take the forms

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \text{ and}
\]

\[
(x - H)^2 + (y - K)^2 = R^2
\]

Substituting the correct values of this example

\[
\frac{(x + 1.8660254)^2}{(2.3660254)^2} - \frac{y^2}{(4.5157839)^2} = 1 \text{ and}
\]

\[
(x - 1.8660254)^2 + y^2 = 3.4820508 \text{ which reduces to}
\]

\[
y^2 = 3.7320508x - x^2
\]
Expanding and substituting the value of $y^2$, the two become 

$$0.22767086x^2 + 0.305020908x - 0.66666685 = 0$$

from which

$$x = 1.1677744$$ \quad and \quad $$y = \pm 1.7304613$$

Returning to

$$\cot \text{Lat} = \frac{x^2 + y^2}{x} \quad \text{and} \quad \tan \text{Lo} = \frac{y}{x}$$

$$\cot \text{Lat} = 2.0876286.$$ 

Lat = 25 deg. 35 min. 42 sec. North

$$\tan \text{Lo} = \pm 1.4818456$$

Lo = 55 deg. 59 min. 15.8 sec. W of meridian of GP.
V. HYPERBOLIC MID-MORNING SIGHT

(HEMISPHERE NOT KNOWN)

It does not matter that the navigator may not know in which hemisphere his position lies. The ship is always in the same hemisphere as the celestial body on which the navigator obtains a right angle sight. If the body is recognized, its declination can be found in the Nautical Almanac or the Ephemeris.

If it is found that the ship is not in the hemisphere at first supposed, the circle of position is simply projected forward and backward to the polar plane, the east-west directions are reversed and the name of the pole is changed. The equations are built up by the method used in all these problems and solved simultaneously, as before. The ship's position is at P or P', depending on whether it was east or west of the celestial body at the time of the sight. The east-west directions on the chart and the poles are interchanged for the solution of this problem.

Example: Alt. 50° Decl. -20° Zn 90° or 270°
GP' = reverse projection of geographical position of body
D = reverse projection of lower intercept of circle of position
CGP' = \( \frac{OGP'}{2} \) = R (radius of circle of equal azimuths)
GP'' = point corresponding to GP' for construction purposes
C'GP' = \( \frac{GP'GP''}{2} \)
C' = intercept of asymptotes h,k.
The equations take the same form as before

\[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{and} \quad (x - H)^2 + (y - K)^2 = R^2
\]

With the correct substitutions for this example these become

\[
\frac{(x - 1.08506555)^2}{(1.328013)^2} - \frac{y^2}{(3.453223)^2} = 1, \quad \text{and}
\]

\[
(x + 1.3737887)^2 + y^2 = 1.887158016 \quad \text{which reduces to}
\]

\[
y^2 = -(x^2 + 2.7474774x)
\]

Expanding the first equation and substituting the value of \(y^2\), the pair reduces to

\[
.445703598x^2 - .5548470871x - .57397779 = 0 \quad \text{from which}
\]

\[x = - .6720925 \quad \text{and} \quad y = \pm 1.1767968
\]

Again returning to the forms

\[
cot \text{ Lat} = \sqrt{\frac{x^2 + y^2}{x}} \quad \text{and} \quad \tan \text{ Lo} = \frac{y}{x}
\]

\[
cot \text{ Lat} = 1.35519702
\]

\[
\text{Lat.} = 36 \text{ deg. } 25 \text{ min. } 34.8 \text{ sec. } \text{S}, \quad \text{since the figure is a reverse projection of the southern hemisphere.}
\]

\[
\tan \text{ Lo} = 1.7509447
\]

\[
\text{Lo} = 60 \text{ deg. } 16 \text{ min. } 06 \text{ sec. West of meridian of GP (at P), as the figure is a reverse projection of the southern hemisphere.}
\]
BIBLIOGRAPHY

This bibliography contains the references which have been especially helpful in the preparation of this manuscript.


